

Some Basics of Distribution Theory

Conditional Distributions

Let X and Y be two random variables.

$$1. f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}.$$

$$2. f_{X,Y}(x,y) = f_{Y|X}(y|x)f_X(x).$$

$$3. f_Y(y) = \begin{cases} \sum_x f_{X,Y}(x,y) & X \text{ discrete} \\ \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx & X \text{ continuous} \end{cases}$$

Example 1

$$f_{X,Y}(x,y) = x + y, \quad 0 < x < 1, 0 < y < 1.$$

What is $f_{Y|X}(y|x)$?

Answer:

$$\begin{aligned} f_X(x) &= \int_0^1 (x+y) dy \\ &= \left[xy + \frac{y^2}{2} \right]_0^1 \\ &= x + \frac{1}{2}, \quad 0 < x < 1. \end{aligned}$$

$$\begin{aligned} \Rightarrow f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \frac{x+y}{x+\frac{1}{2}}, \\ &= \frac{2(x+y)}{2x+1}, \quad 0 < x < 1, 0 < y < 1. \end{aligned}$$

Example 2

$$f_{Y|X}(y|x) = \frac{1}{x}, \quad 0 < y < x.$$

$$f_X(x) = \frac{1}{5}, \quad 0 < x < 5.$$

What is $f_Y(y)$?

Answer:

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx \\ &= \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x) dx \\ &= \int_y^5 \frac{1}{x} \cdot \frac{1}{5} dx \\ &= \frac{1}{5} \left[\ln(x) \right]_y^5 \\ &= \frac{\ln(5) - \ln(y)}{5}, \quad 0 < y < 5 \end{aligned}$$

Class Exercise

$$f_{X,Y}(x,y) = \frac{6(x^2 + 3y)}{11},$$
$$0 < x < 1, 0 < y < 1.$$

Determine $f_{Y|X}(y|x)$.

Answer:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy \\ &= \int_0^1 \frac{6x^2 + 18y}{11} dy \\ &= \left[\frac{6x^2 y + 9y^2}{11} \right]_0^1 \\ &= \frac{6x^2 + 9}{11}, \quad 0 < x < 1. \end{aligned}$$

$$\text{i.e. } f_X(x) = \frac{6x^2 + 9}{11}, \quad 0 < x < 1.$$

Then

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{6(x^2 + 3y)}{11} \bigg/ \frac{6x^2 + 9}{11} \\ &= \frac{2(x^2 + 3y)}{2x^2 + 3} \\ &\quad \text{for } 0 < x < 1, \quad 0 < y < 1. \end{aligned}$$

