UNIVERSITY OF TECHNOLOGY SYDNEY

School of Mathematical and Physical Sciences

37457 Advanced Bayesian Methods

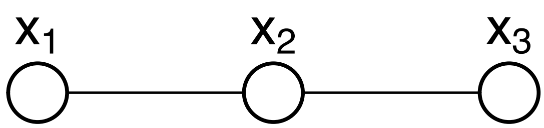
ASSIGNMENT 3

Due time and date: 10:05am, Wednesday 4th September 2024.
Submission method:. Hand to Professor Wand at the start of class.

NOTES:

- All working should be shown.
- There are no group assignments or laboratories in 37457 Advanced Bayesian Methods. All assessment tasks require individual work only.
- For the benefit of participants requiring assistance with this assignment, a help session will be held at 3pm-4pm on Tuesday 3rd September 2024 in Room 006, Level 6, Building 7.

1. (a) Let $p(x_1, x_2, x_3)$ be a joint density function of three continuous random variables x_1, x_2 and x_3 such that the undirected graph shown below is a probabilistic undirected graph with respect to p.



Explain, using an appropriate result in Section 2.6 of the *Graph Theory and Statistics* notes, why

$$x_1 \perp\!\!\!\perp x_3 | x_2. \tag{1}$$

(b) In this question, the goal is to show from first principles that $x_1 \perp\!\!\!\perp x_3 | x_2$ for the above undirected graph. Therefore, result (1) should not be assumed. Instead it is derived using the *potential function*-based definition of $p(x_1, x_2, x_3)$. The general form of a density function respected by the above probabilistic graph is

$$p(x_1, x_2, x_3) = \psi_{12}(x_1, x_2)\, \psi_{23}(x_2, x_3)/C$$

where ψ_{12} and ψ_{13} are potential functions and

$$C \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_{12}(x_1, x_2)\, \psi_{23}(x_2, x_3)\, dx_1\, dx_2\, dx_3$$

is the normalising factor. Show that

$$p(x_1, x_3 | x_2) = \left\{ \frac{\psi_{12}(x_1, x_2)}{\int_{-\infty}^{\infty} \psi_{12}(x_1, x_2)\, dx_1} \right\} \left\{ \frac{\psi_{23}(x_2, x_3)}{\int_{-\infty}^{\infty} \psi_{23}(x_2, x_3)\, dx_3} \right\}.$$

Similar calculations (which you are not required to do as part of this assignment) can be used to show that

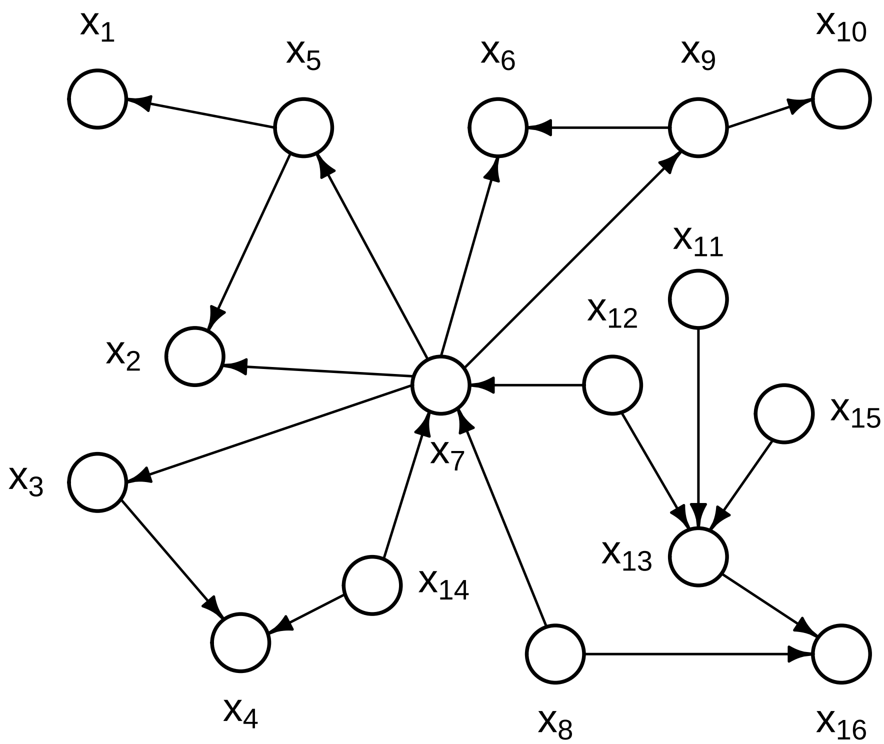
$$p(x_1|x_2) = \frac{\psi_{12}(x_1, x_2)}{\int_{-\infty}^{\infty} \psi_{12}(x_1, x_2)\, dx_1} \qquad \text{and} \qquad p(x_3|x_2) = \frac{\psi_{23}(x_2, x_3)}{\int_{-\infty}^{\infty} \psi_{23}(x_2, x_3)\, dx_3}$$

which leads to

$$p(x_1, x_3|x_2) = p(x_1|x_2)p(x_3|x_2)$$

and confirms result (1).

2. Consider the following probabilistic DAG:



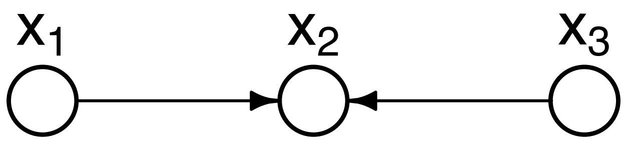
with corresponding joint density function $p(x_1, \ldots, x_{16})$.

(a) Draw the smallest ancestral sub-graph containing $\{x_1, x_4, x_{12}, x_{13}, x_{15}\}$.

(b) Draw the moral graph of your answer to part (a).

(c) Is it true that $\{x_1, x_4\} \perp\!\!\!\perp \{x_{13}, x_{15}\}|x_{12}$? Give reasons for your answer.

3. Consider the probabilistic DAG:



and suppose that

$$x_2|x_1, x_3 \sim N\big(3\,x_1 + 5, 1/(8x_3)\big), \quad x_1 \sim N\big(13, \tfrac{1}{16}\big), \quad x_3 \sim \text{Gamma}(10, 1). \qquad (2)$$

Determine $p(x_3|x_1, x_2)$ – known as the *full conditional* density function of x_3 with respect to the graph.

Hints:

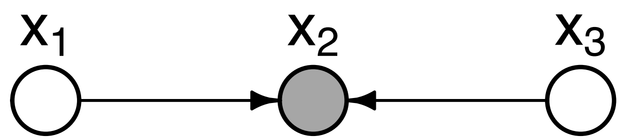
- Use a proportionality argument where factors not involving x_3 can be ignored.
- See the examples in Section 2.9.1 of the *Graph Theory and Statistics* notes.

4. For the probabilistic DAG in the previous question and distributions given by (2) determine $p(x_1|x_2, x_3)$ – known as the *full conditional* density function of x_1 with respect to the graph.

   Hints:

   - Use a proportionality argument where factors not involving x_1 can be ignored.
   - Use Result 2.5 in the *Graph Theory and Statistics* notes.
   - See the examples in Section 2.9.1 of the *Graph Theory and Statistics* notes.

5. Consider the probabilistic DAG from the previous two questions, but now suppose that x_2 is observed as data:



   Use your answers from the two previous questions to write down a Markov chain Monte Carlo algorithm for drawing samples from

$$p(x_1|x_2 = \mathring{x}_2) \quad \text{and} \quad p(x_3|x_2 = \mathring{x}_2).$$

   Hint: See the examples in Section 2.9.1 of the *Graph Theory and Statistics* notes.

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