35007

Uniformly and Lipschitz Continuous Functions

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Function Continuity Ranking

Lipschitz

IMPLIES

Uniform

IMPLIES

Ordinary (or "Pointwise")

Function Continuity Ranking (Alternative)

Ordinary (or "Pointwise")

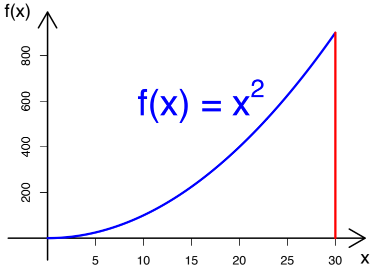
IS WEAKER THAN

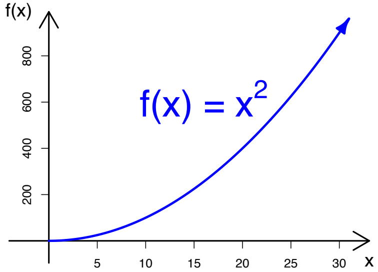
Uniform

IS WEAKER THAN

Lipschitz

Uniform Continuity by Examples





uniformly contin. on $[0, 30]$

NOT uniformly contin. on $[0, \infty)$

Algebraic Reinforcement

$f(x) = x^2$ is

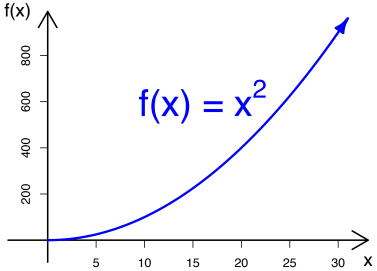
uniformly continuous on $[0, 30]$,

uniformly continuous on $[0, 10^{100}]$,

NOT uniformly continuous on $[0, \infty]$,

NOT uniformly continuous on $\mathbb{R}$.

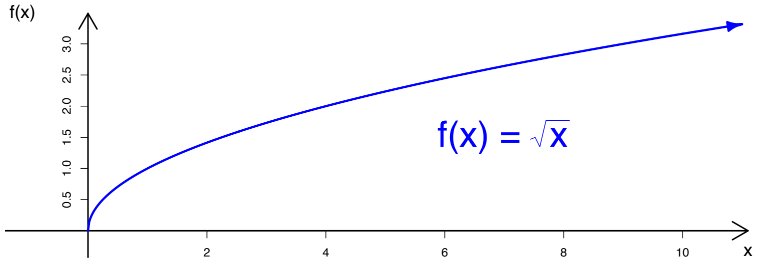
Intuition



As we move towards ∞, a same-sized change in x leads to arbitrarily large and unbounded changes in $f(x)$

$\Longrightarrow$ continuity is NOT uniform.

$\sqrt{x}$ instead of x^2

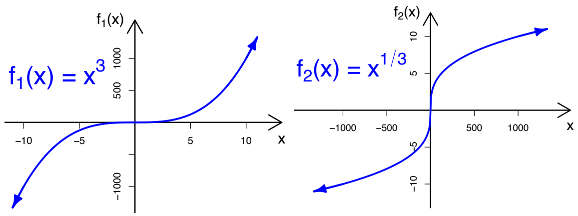


As we move towards ∞, a same-sized change in x leads to RELATIVELY SMALL changes in $f(x)$

$\Longrightarrow$ continuity IS uniform.

Class Quick Quiz

***YES* or *NO* for uniformly continuous?**



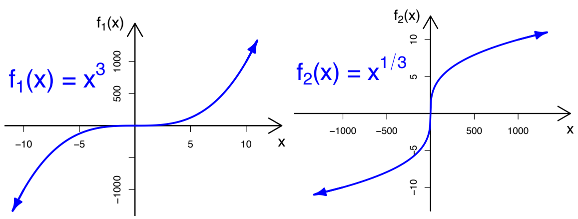
[1.] $f_1(x) = x^3$ on $\mathbb{R}$ [2.] $f_2(x) = x^{1/3}$ on $\mathbb{R}$

[3.] $f_3(x) = x^{\text{googol}}$ on $[-\text{googol}, \text{googol}]$ where

$\text{googol} = 10^{100}$.

Asymptote Examples

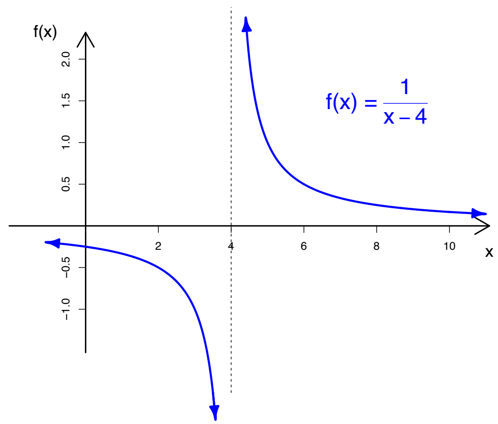
Class Quick Quiz SOLUTIONS



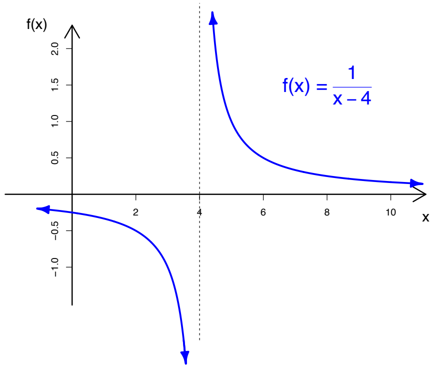
[1.] $f_1(x) = x^3$ on $\mathbb{R}$ ***NO*** [2.] $f_2(x) = x^{1/3}$ on $\mathbb{R}$ ***YES***

[3.] $f_3(x) = x^{\text{googol}}$ on $[-\text{googol}, \text{googol}]$ ***YES***

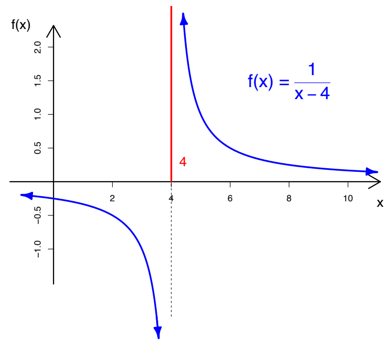
$$f(x) = \frac{1}{x-4}, \quad x \neq 4.$$



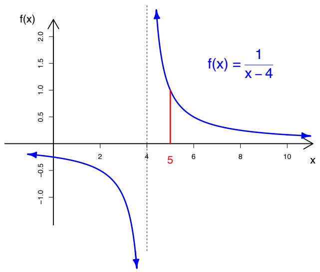
$$f(x) = \frac{1}{x-4}, \quad x \neq 4.$$



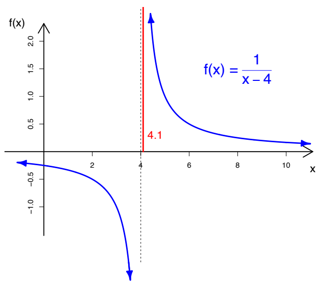
- not pointwise continuous on $\mathbb{R}$,
- but IS pointwise continuous on $(4, \infty)$.



NOT uniformly continuous on $(4, \infty)$

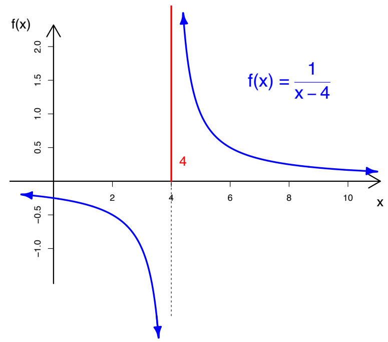


unif. contin. on $[5, \infty)$



unif. contin. on $[4.1, \infty)$

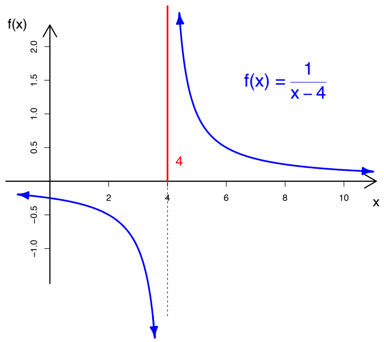
Intuition



As we move towards 4 from the right, small changes in x lead to arbitrarily large and unbounded changes in $f(x)$

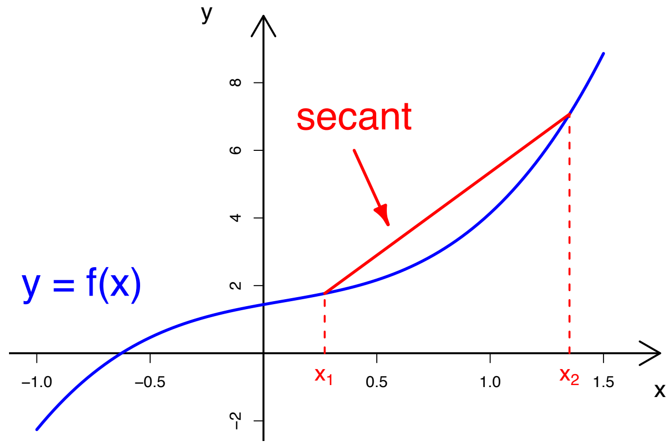
$\Longrightarrow$ continuity is NOT uniform.

Proving NOT Uniformly Continuous



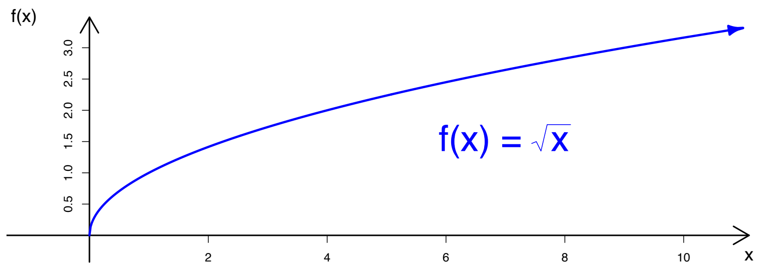
Exercise F.4 is concerned with the relevant maths.

FIRST: Definition of Secant



Even Stronger Continuity

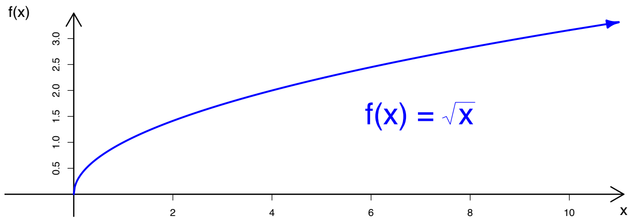
The Lipschitz Additional Stringency



$f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$

BUT $f(x) = \sqrt{x}$ is NOT Lipschitz continuous on $[0, \infty)$

Intuition



When approaching zero from the right, the secant gradients

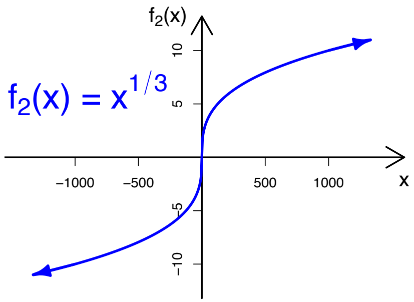
$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{\sqrt{x_2}-\sqrt{x_1}}{x_2-x_1}, \quad 0 \le x_1 < x_2$$

become arbitrarily steep

$\Longrightarrow$ continuity is NOT Lipschitz.

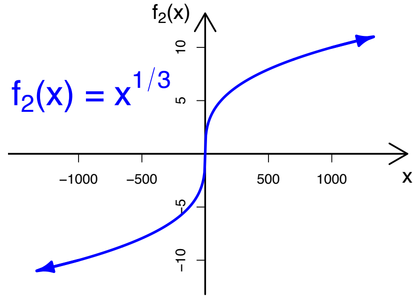
Class Quick Quiz

Is $f_2(x) = x^{1/3}$ Lipschitz continuous on $\mathbb{R}$?



Class Quick Quiz SOLUTION

Is $f_2(x) = x^{1/3}$ Lipschitz continuous on $\mathbb{R}$?



The answer is. . . NO, IT AIN'T.

Uniform and Lipschitz Continuity Proofs

Uniform and Lipschitz Continuity Proofs

Learn by doing (the exercises).

The Most Relevant Exercises
(from the current subject exercise sets)

Exercise Set F

♦ ♦ ♦