35007

Uniformly and Lipschitz Continuous Functions

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Function Continuity Ranking

Lipschitz

IMPLIES

Uniform

IMPLIES

Ordinary (or “Pointwise”)

Function Continuity Ranking (Alternative)

Ordinary (or “Pointwise”)

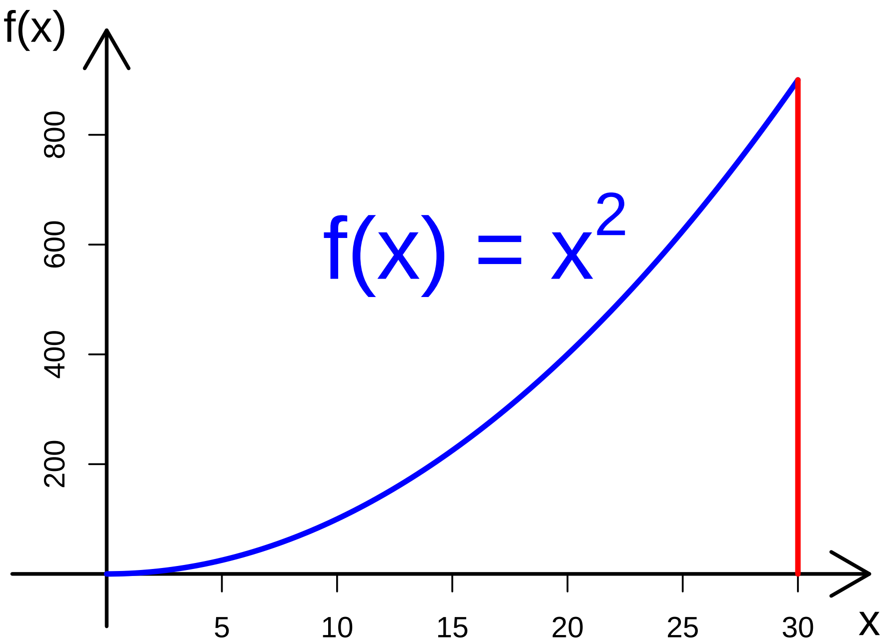
IS WEAKER THAN

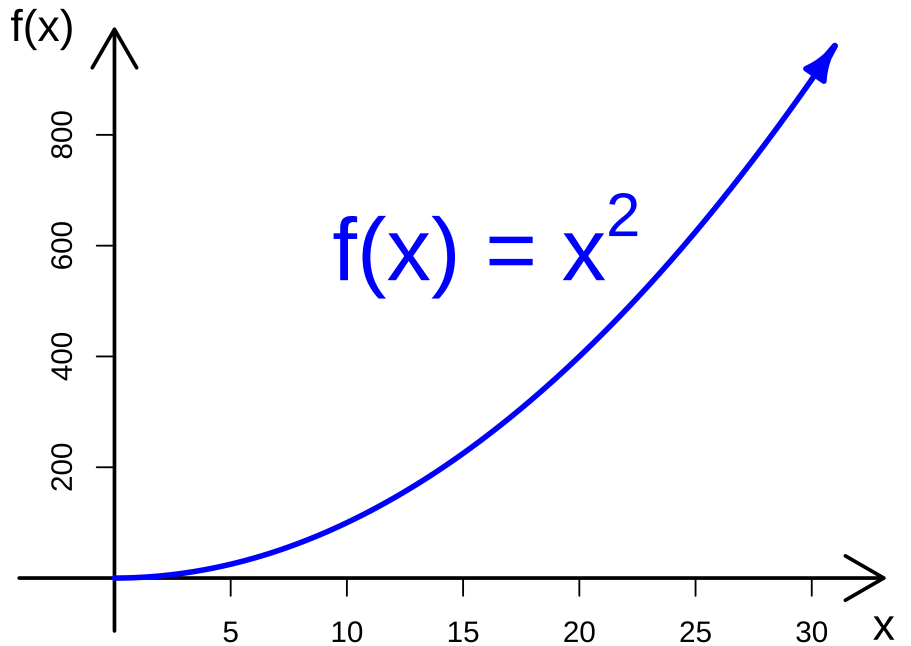
Uniform

IS WEAKER THAN

Lipschitz

Uniform Continuity by Examples

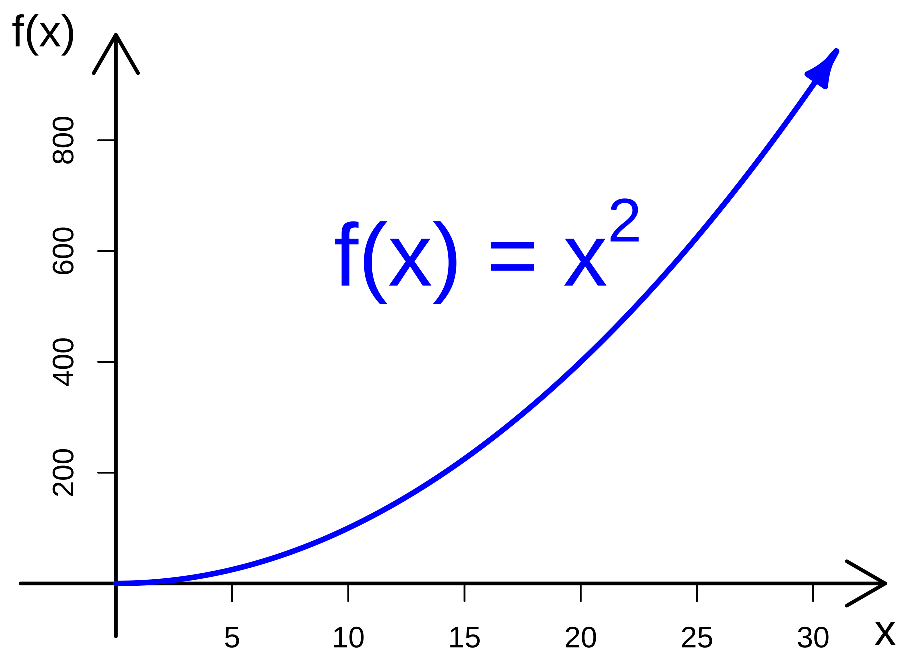




uniformly contin.
on $[0, 30]$

NOT uniformly contin.
on $[0, \infty)$

Intuition



As we move towards ∞, a same-sized change in x leads to arbitrarily large and unbounded changes in $f(x)$

$\Longrightarrow$ continuity is NOT uniform.

Algebraic Reinforcement

$f(x) = x^2$ is

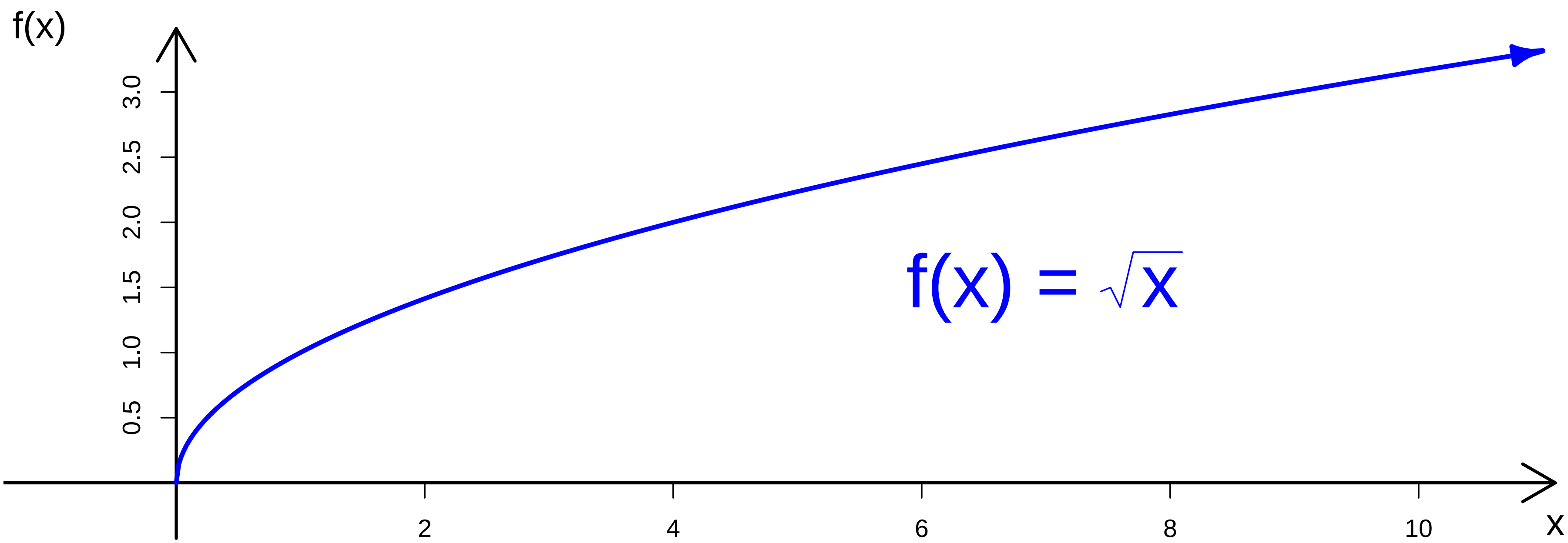
uniformly continuous on $[0, 30]$,

uniformly continuous on $[0, 10^{100}]$,

NOT uniformly continuous on $[0, \infty]$,

NOT uniformly continuous on $\mathbb{R}$.

$\sqrt{x}$ *instead of* x^2

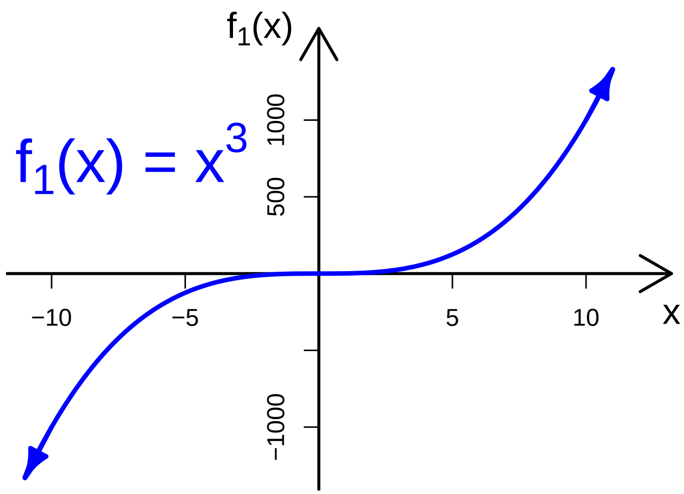


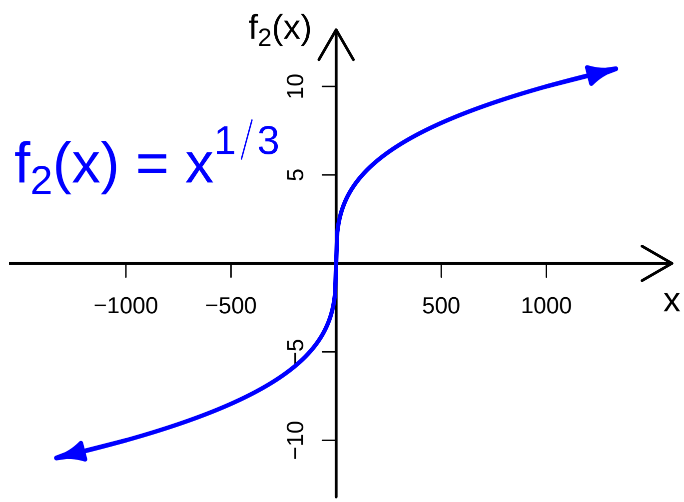
As we move towards ∞, a same-sized change in x leads to RELATIVELY SMALL changes in $f(x)$

$\implies$ continuity IS uniform.

Class Quick Quiz

YES ***or*** ***NO*** ***for uniformly continuous?***





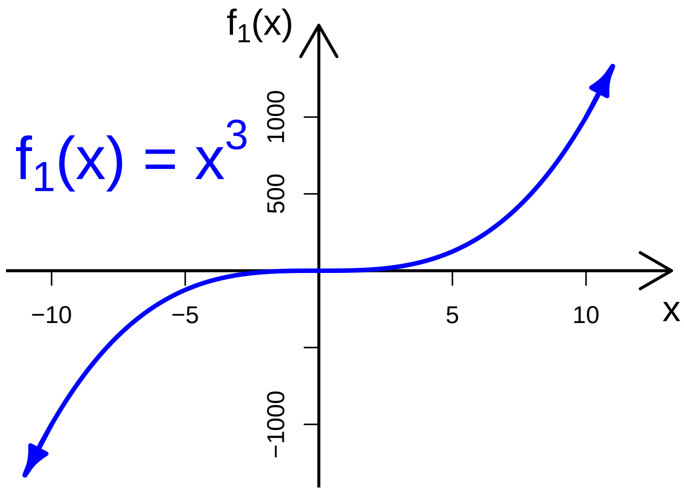
[1.] $f_1(x) = x^3$ on $\mathbb{R}$

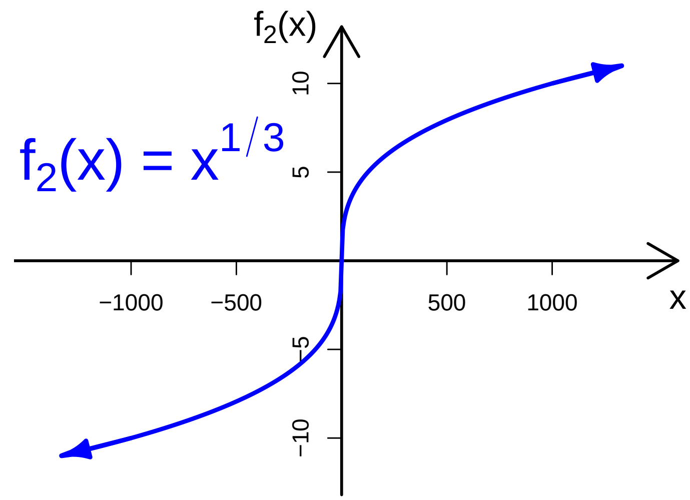
[2.] $f_2(x) = x^{1/3}$ on $\mathbb{R}$

[3.] $f_3(x) = x^{\text{googol}}$ on $[-\text{googol}, \text{googol}]$ where

$$\text{googol} = 10^{100}.$$

Class Quick Quiz SOLUTIONS



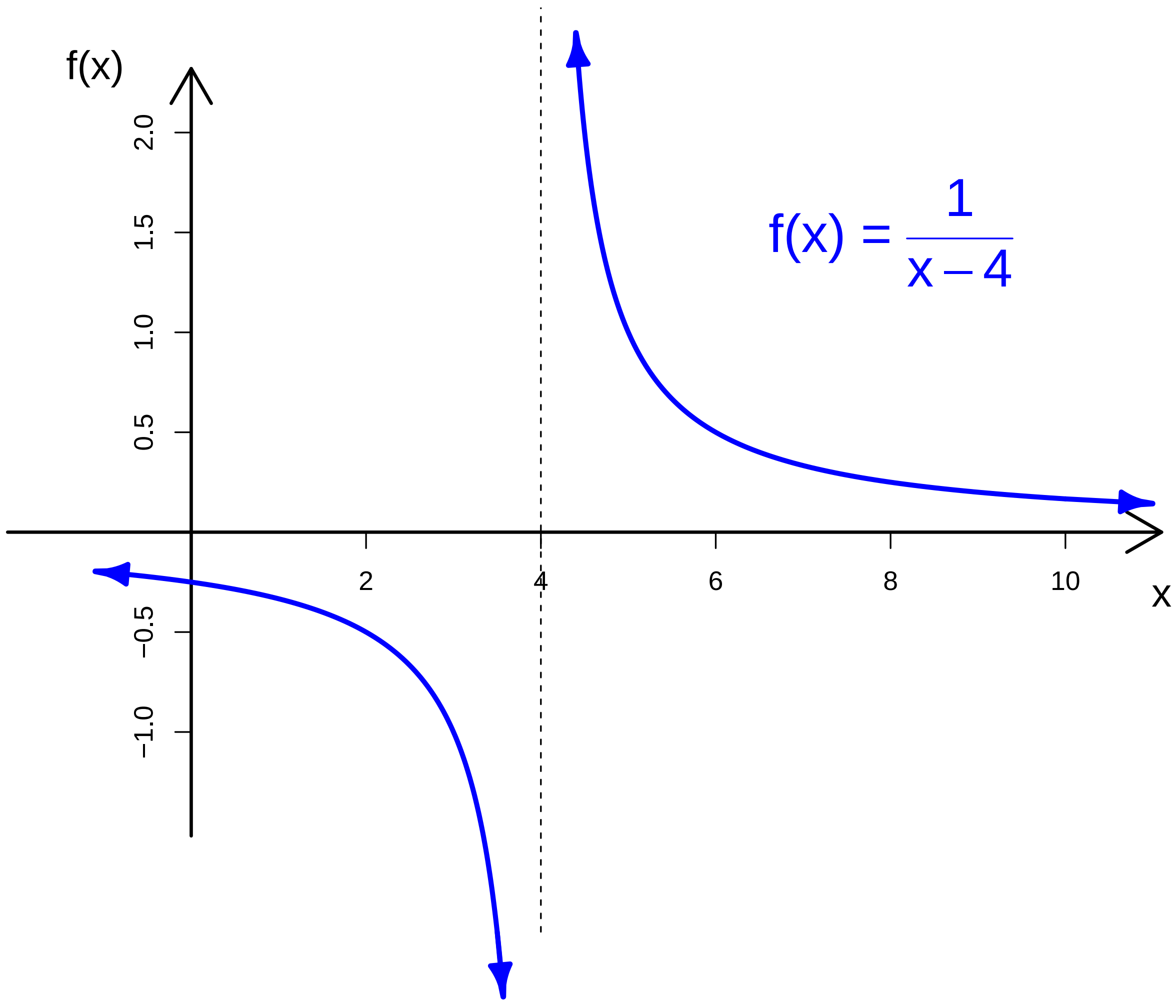


[1.] $f_1(x) = x^3$ on $\mathbb{R}$ ***NO*** [2.] $f_2(x) = x^{1/3}$ on $\mathbb{R}$ ***YES***

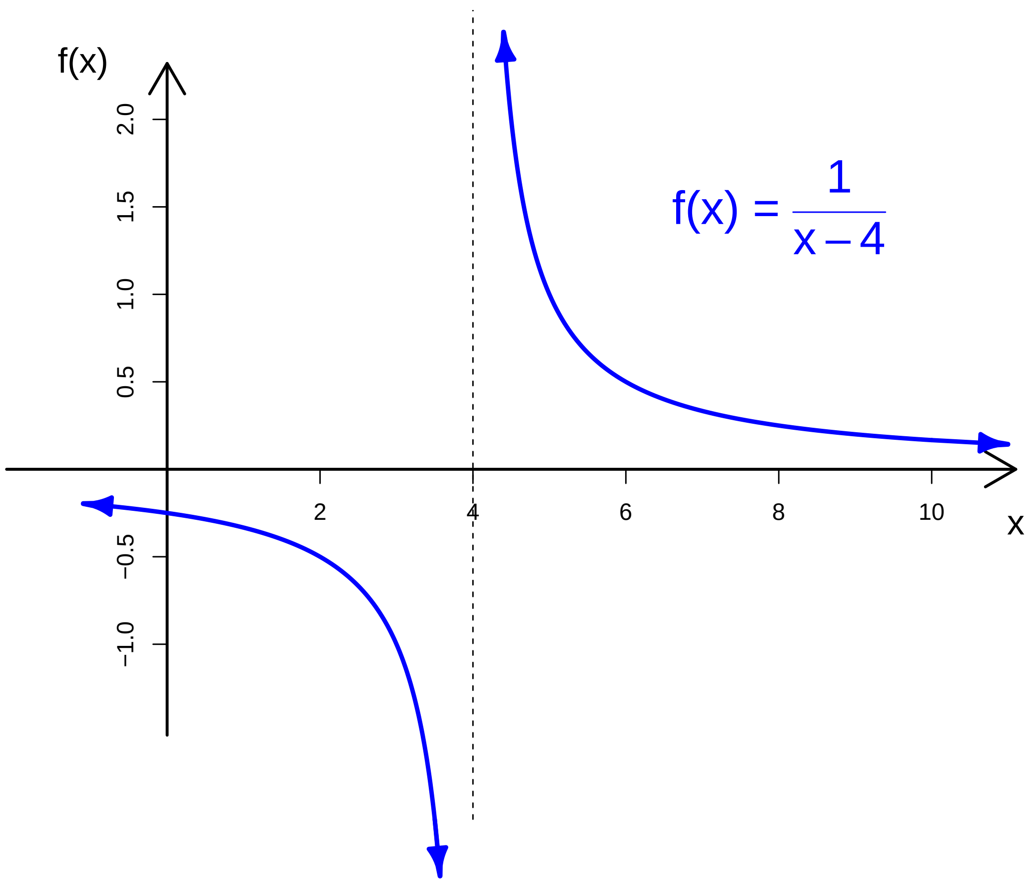
[3.] $f_3(x) = x^{\text{googol}}$ on $[-\text{googol}, \text{googol}]$ ***YES***

Asymptote Examples

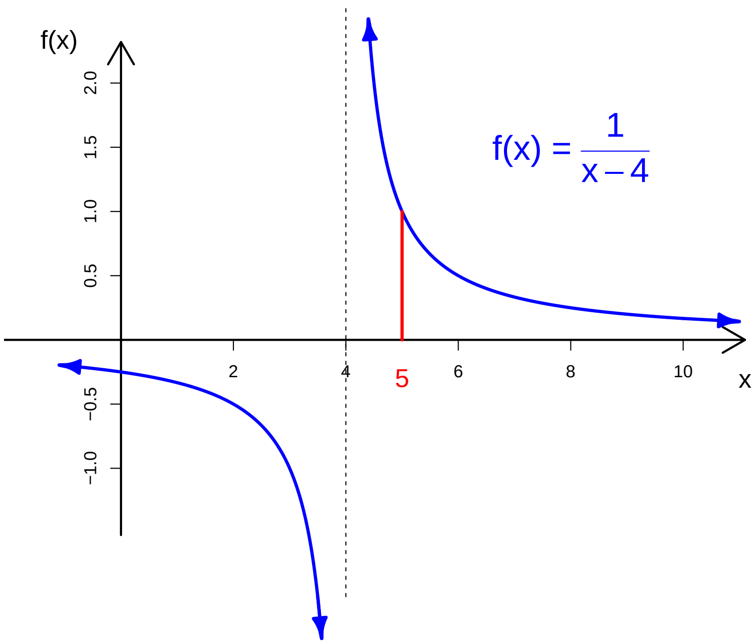
$$f(x) = \frac{1}{x - 4}, \quad x \neq 4.$$



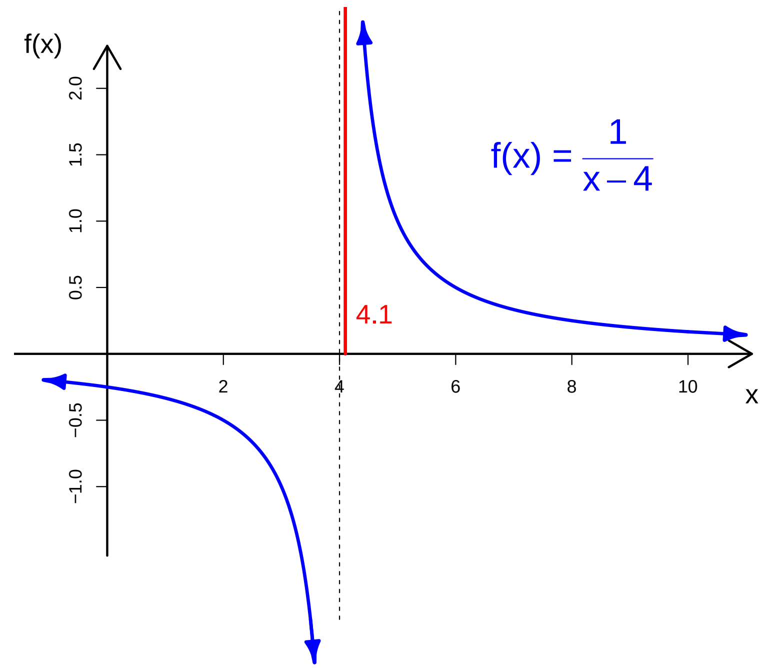
$$f(x) = \frac{1}{x-4}, \quad x \neq 4.$$



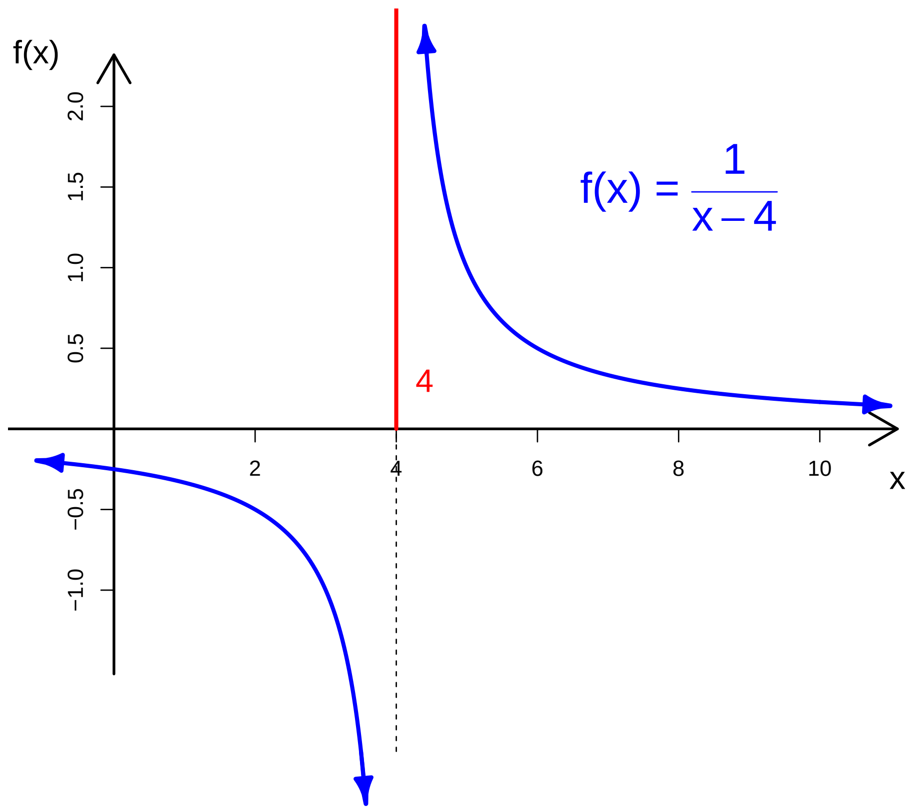
- not pointwise continuous on $\mathbb{R}$,
- but IS pointwise continuous on $(4, \infty)$.



unif. contin. on $[5, \infty)$

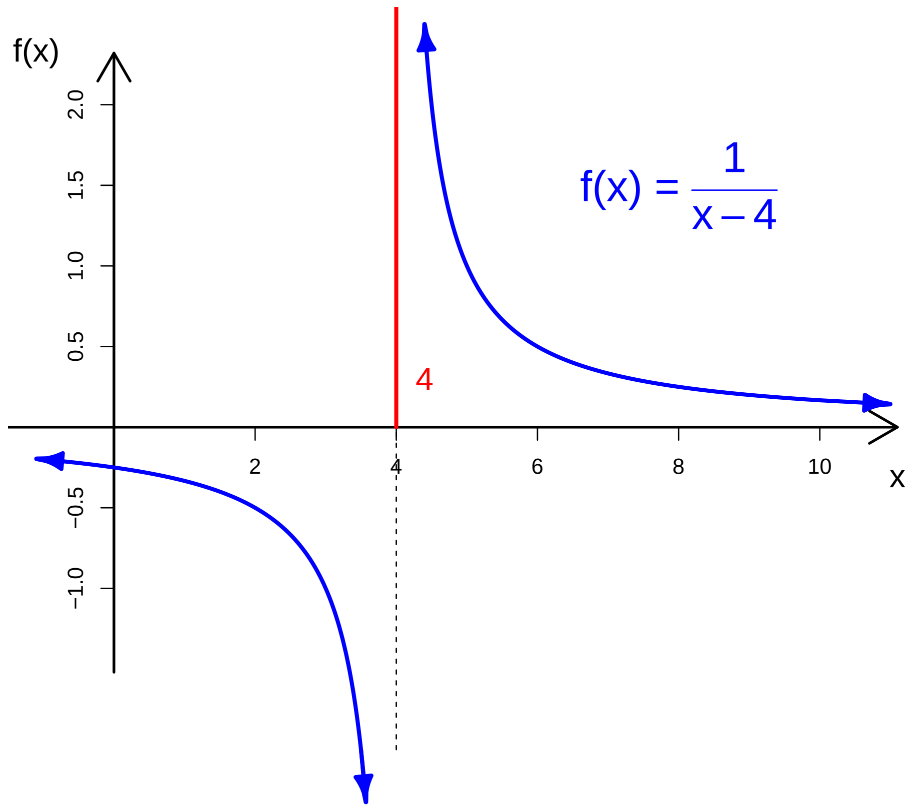


unif. contin. on $[4.1, \infty)$



NOT uniformly continuous on $(4, \infty)$

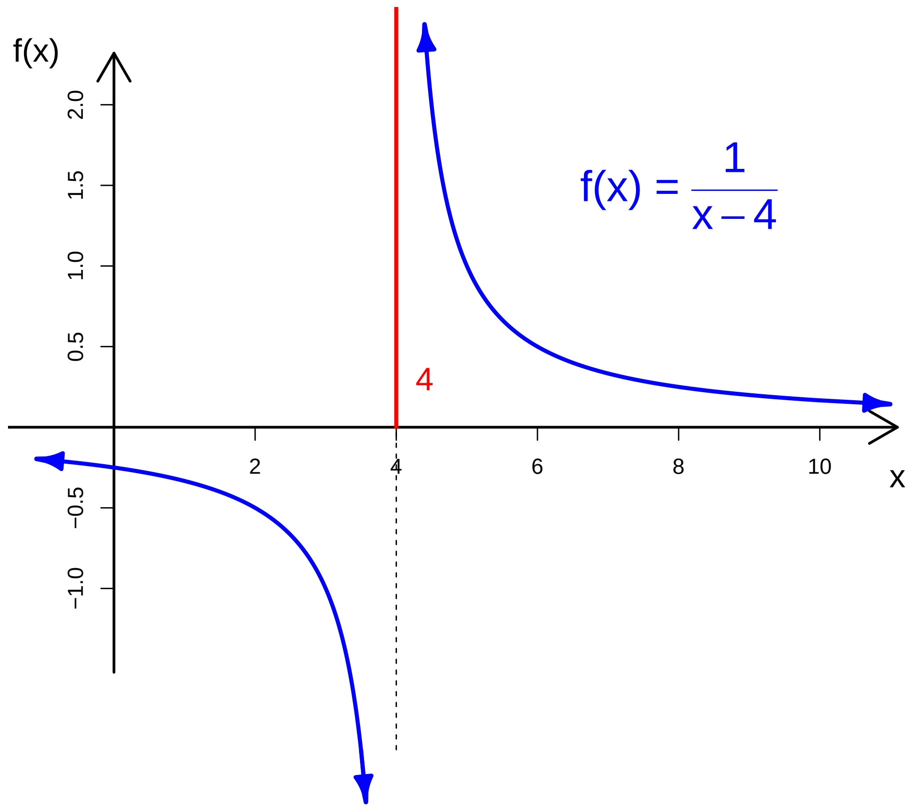
Intuition



As we move towards 4 from the right, small changes in x lead to arbitrarily large and unbounded changes in $f(x)$

$\Longrightarrow$ continuity is NOT uniform.

Proving NOT Uniformly Continuous

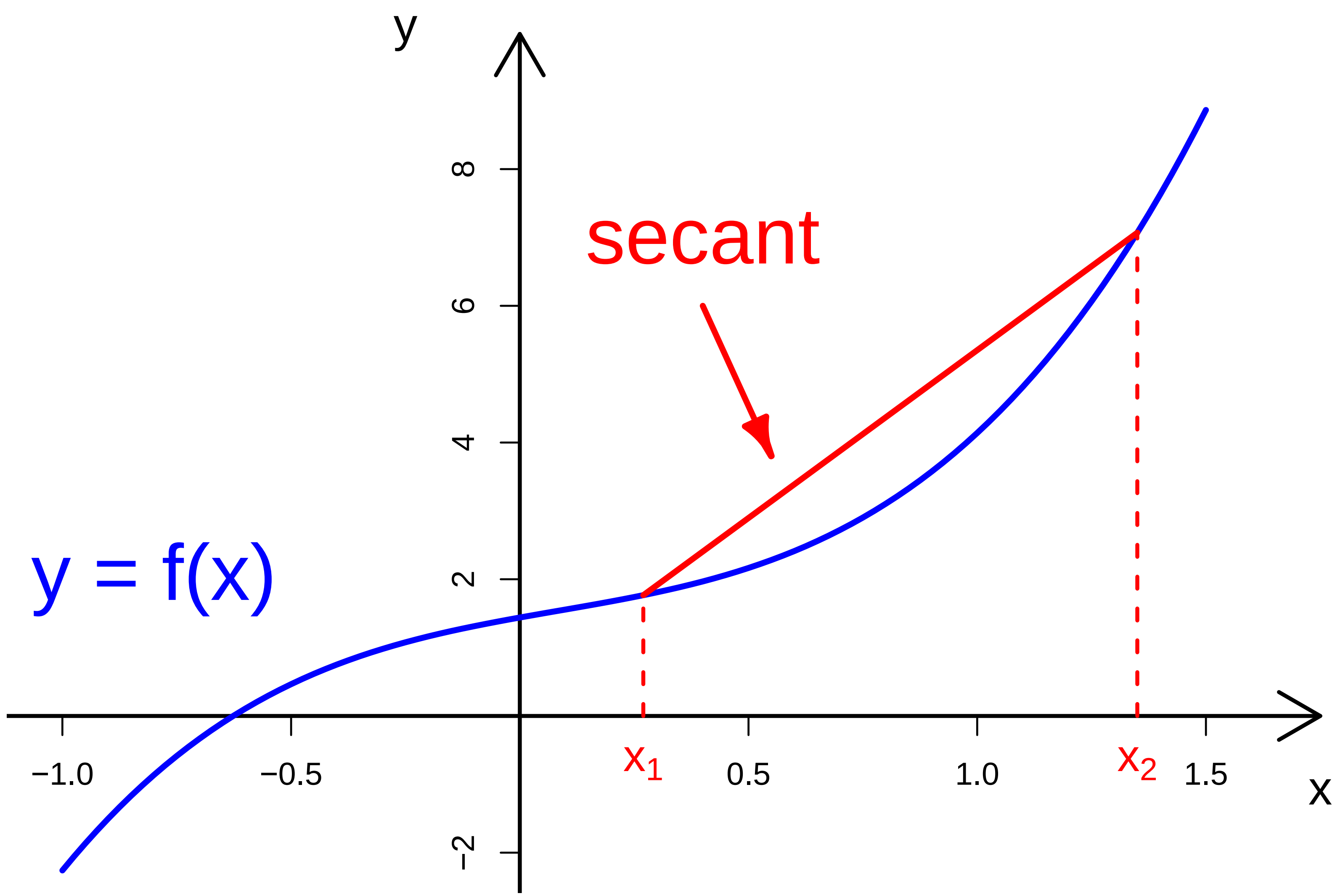


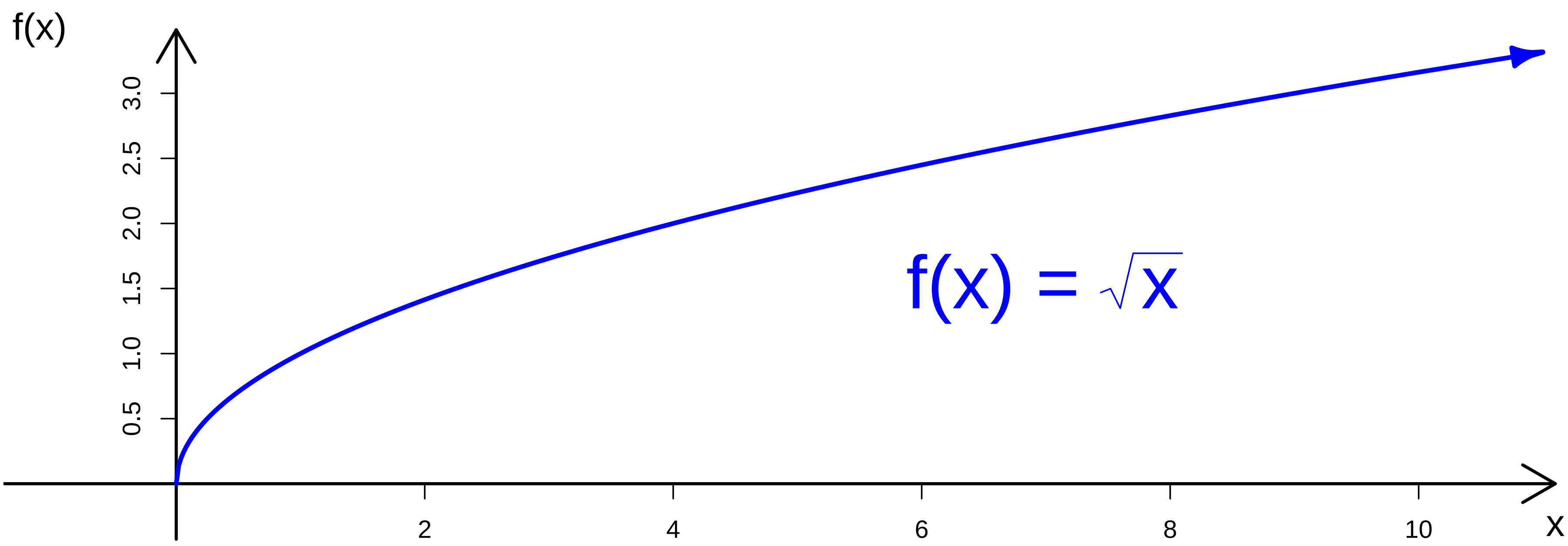
Exercise F.4 is concerned with the relevant maths.

Even Stronger Continuity

The Lipschitz Additional Stringency

FIRST: Definition of Secant

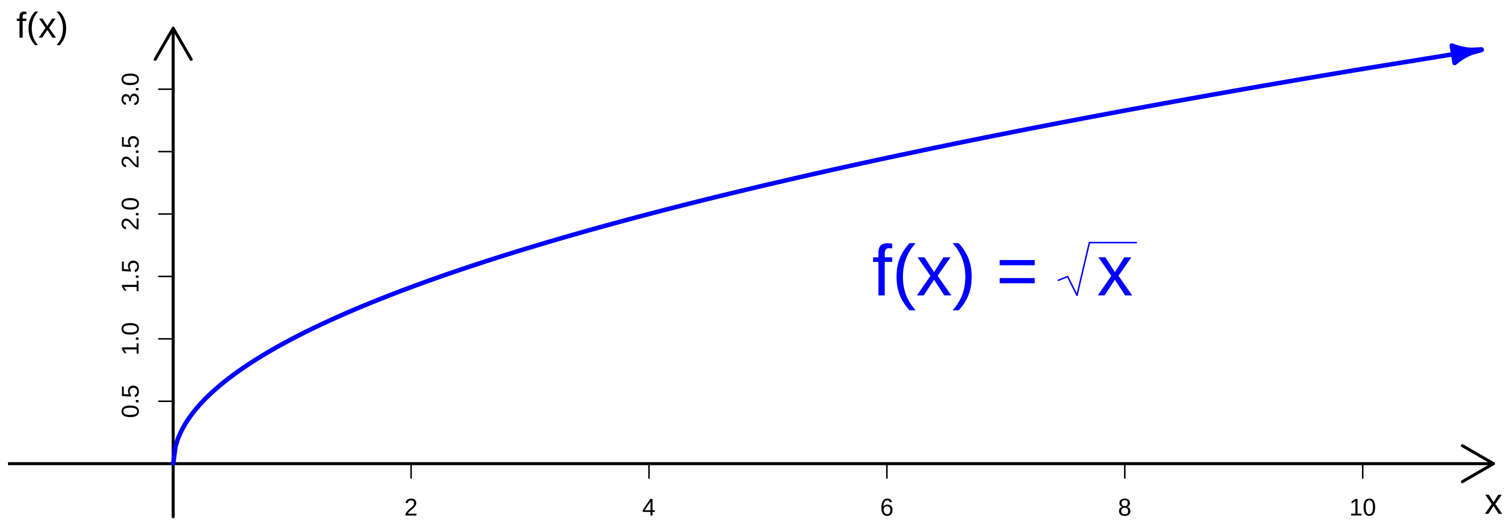




$f(x) = \sqrt{x}$ ***is uniformly continuous on*** $[0, \infty)$

BUT $f(x) = \sqrt{x}$ ***is NOT Lipschitz continuous on*** $[0, \infty)$

Intuition



When approaching zero from the right, the secant gradients

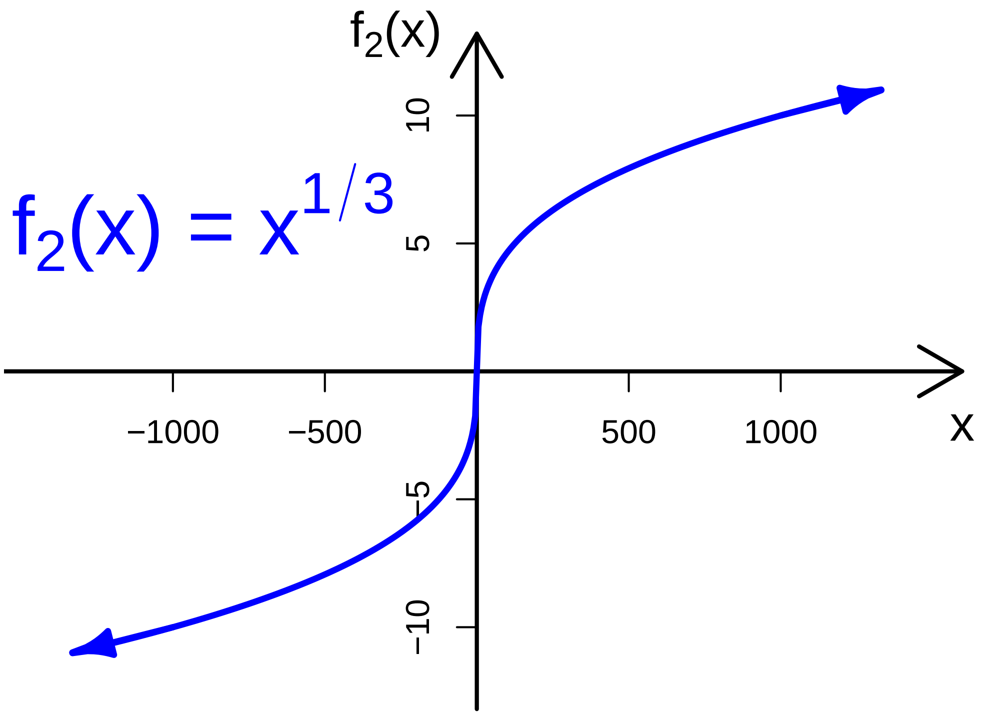
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{\sqrt{x_2} - \sqrt{x_1}}{x_2 - x_1}, \quad 0 \leq x_1 < x_2$$

become arbitrarily steep

$\implies$ continuity is NOT Lipschitz.

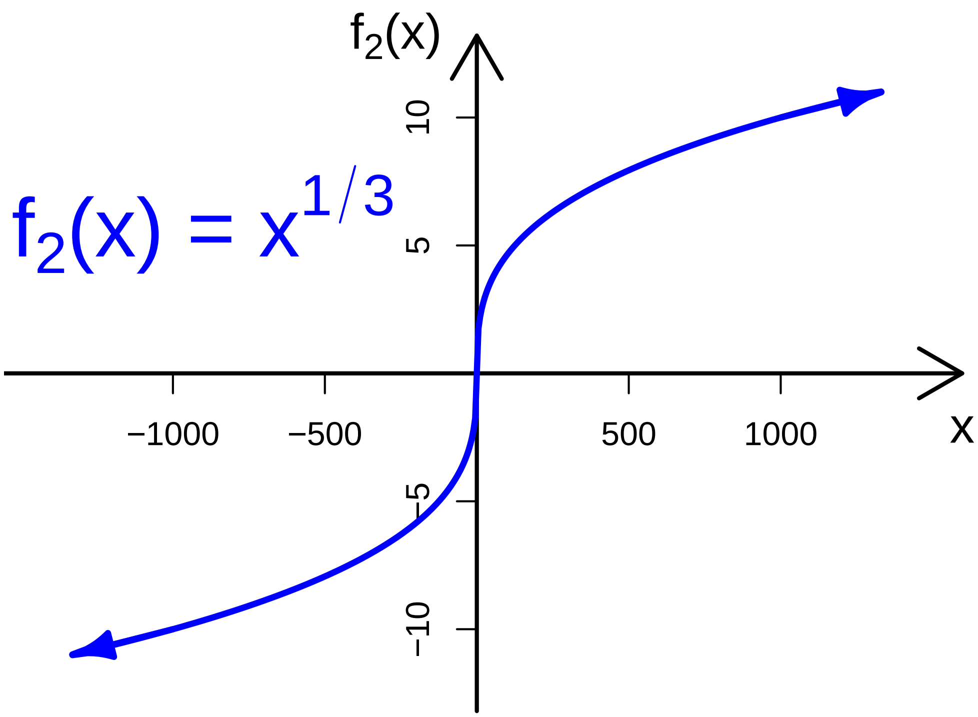
Class Quick Quiz

Is $f_2(x) = x^{1/3}$ ***Lipschitz continuous on*** $\mathbb{R}$ ***?***



Class Quick Quiz SOLUTION

Is $f_2(x) = x^{1/3}$ Lipschitz continuous on $\mathbb{R}$?



The answer is. . . NO, IT AIN'T.

Uniform and Lipschitz Continuity Proofs

Uniform and Lipschitz Continuity Proofs

Learn by doing (the exercises).

The Most Relevant Exercises

(from the current subject exercise sets)

Exercise Set F

♦ ♦ ♦