

35007

Uniform Convergence

Professor Matt Wand

University of Technology Sydney

Key Aspects of Uniform Convergence

- *Concerned with convergence of sequences of functions over subsets of $\mathbb{R}$.*
i.e. “ f_n converges uniformly to f on $X \subseteq \mathbb{R}$ ”
- *Is stronger than pointwise convergence.*
- *Is important for upcoming results about interchanging limits with integrals.*
- *Is not connected at all with uniform continuity – despite sharing the word “uniform”.*

Pointwise Convergence

$$f_n : X \rightarrow \mathbb{R}, \quad n \in \mathbb{N}.$$

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \quad \text{for each } x \in X.$$

We say

f_n converges pointwise to f on X .

Uniform

Convergence

Recommended Replacement of Definition with Supplement Theorem

*Your lecturer recommends replacing
Definition 5.2 for
 f_n converges uniformly to f on $X \subseteq \mathbb{R}$
by*

THEOREM S.2. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions on a set $X \subseteq \mathbb{R}$. Then f_n converges uniformly to the function f on X if and only if*

$$\lim_{n \rightarrow \infty} \sup \{ |f_n(x) - f(x)| : x \in X \} = 0.$$

Useful / intuitive verbal statement

f_n converges uniformly to f on $X \subseteq \mathbb{R}$

if and only if

*the maximum vertical gap
between f_n and f goes to zero.*

Illustrative Examples

$$g_n(x) = x + 1/n, \quad g(x) = x, \quad x \in (0, 1).$$

$$h_n(x) = x^n, \quad h(x) = 0, \quad x \in (0, 1).$$

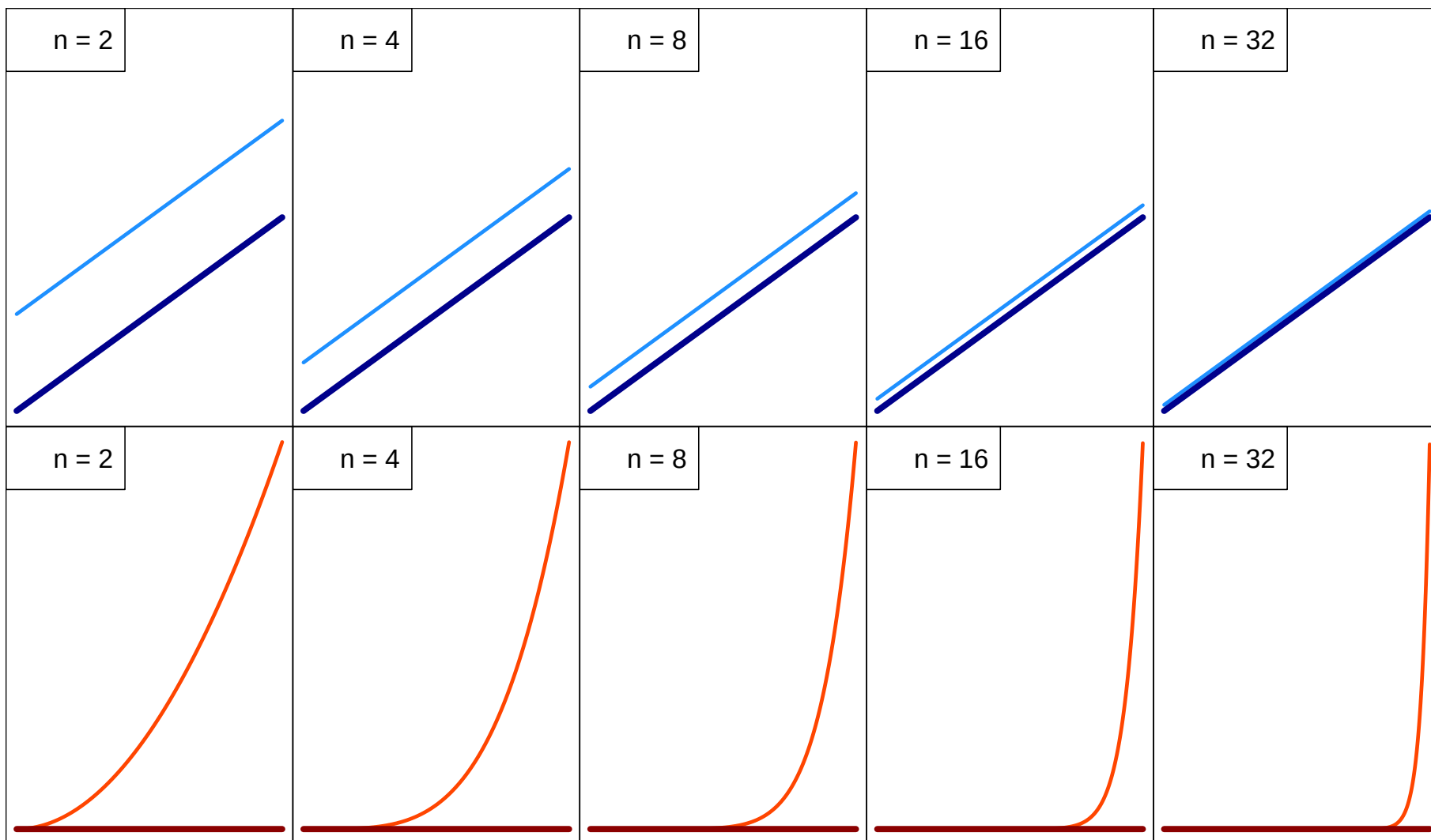
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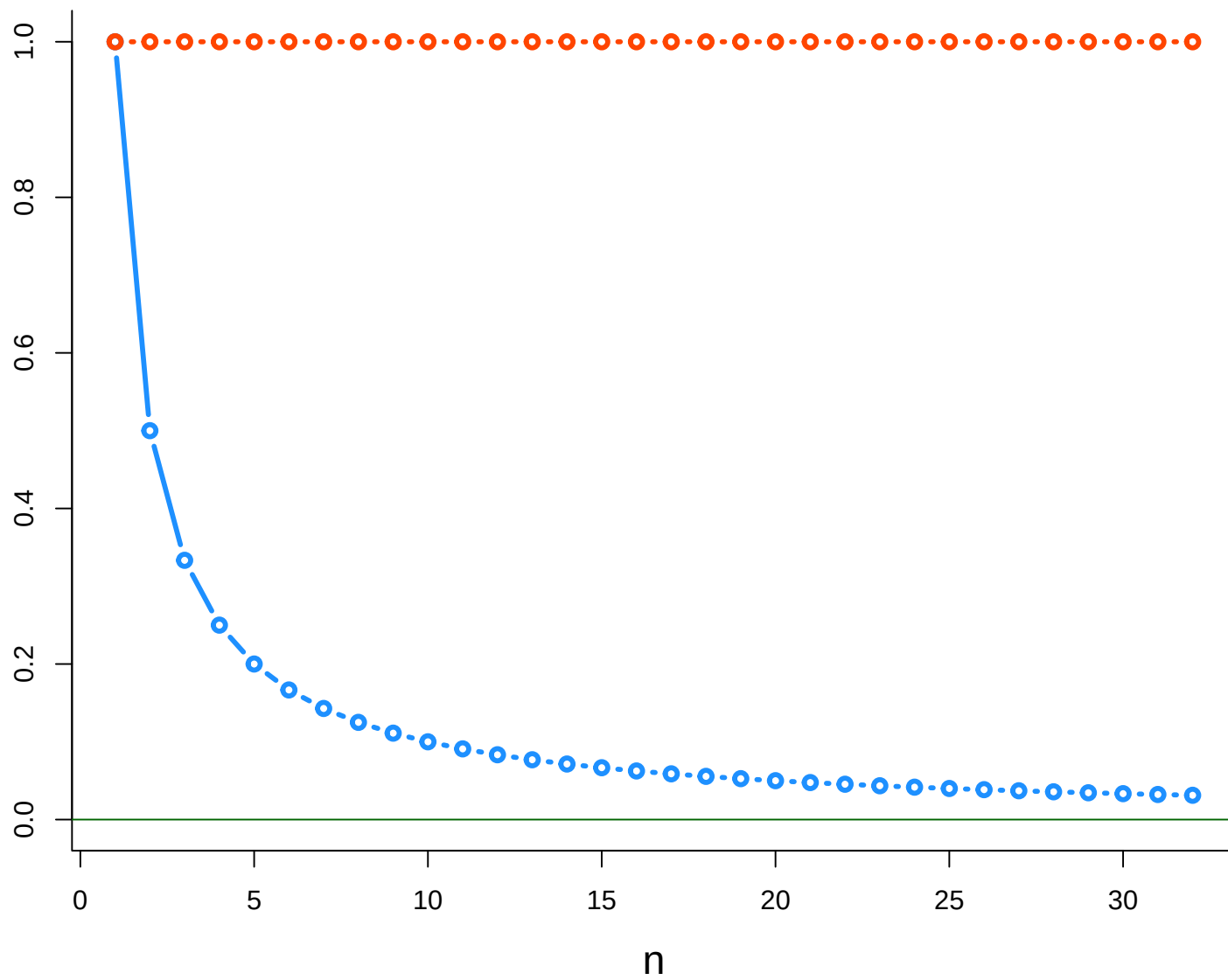
$$h_n(x) = x^n, \quad h(x) = 0, \quad x \in (0, 1).$$

Pointwise results for all $0 < x < 1$:

$$\lim_{n \rightarrow \infty} g_n(x) = g(x) \quad \text{and} \quad \lim_{n \rightarrow \infty} h_n(x) = h(x).$$



sup of vertical gap between sequence and convergent



Illustrative Examples

CONCLUSIONS

g_n converges uniformly to g on $(0, 1)$

h_n DOESN'T converge uniformly to h on $(0, 1)$

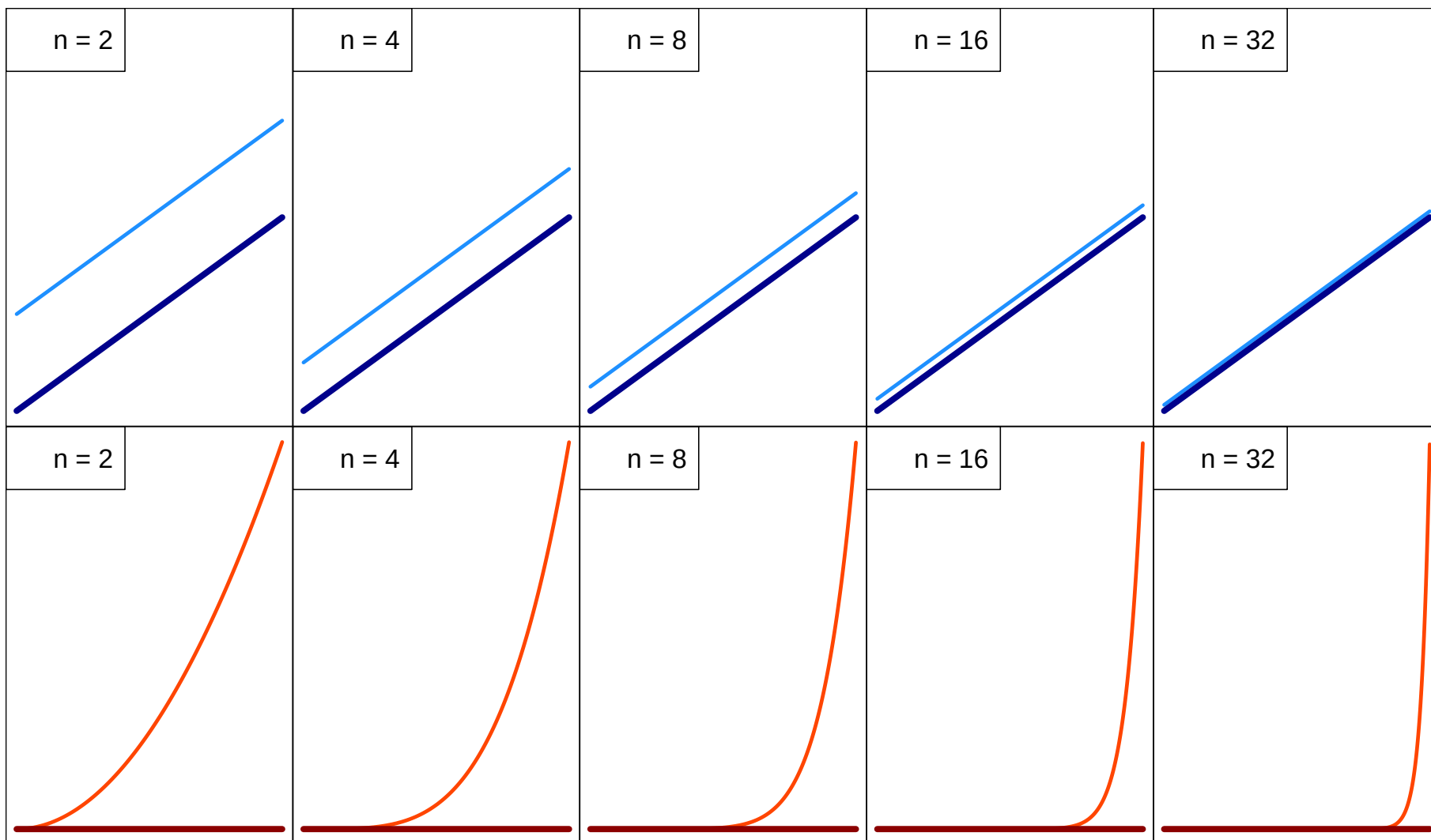
Illustrative Examples

CONCLUSIONS

g_n converges uniformly to g on $(0, 1)$

h_n DOESN'T converge uniformly to h on $(0, 1)$

[even though we have pointwise convergence in
both cases]



Uniform Convergence

has NO connection with

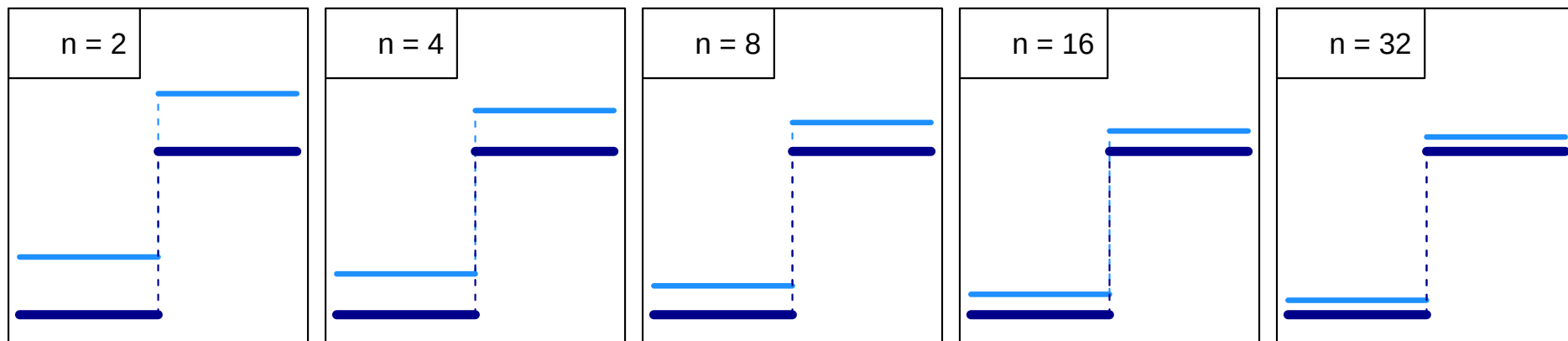
Uniform Continuity

Let

$$\text{step}(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases}$$

and then consider sequence of functions

$$\text{step}(x) + \frac{1}{2\sqrt{n}}, \quad n \in \mathbb{N}, \quad x \in \mathbb{R}.$$



$\text{step} + \frac{1}{2\sqrt{n}}$ **converges uniformly to** step **on** $\mathbb{R}$

despite none of the functions being continuous on $\mathbb{R}$.

Nowhere ***continuous functions***

BUT

uniform convergence

Nowhere continuous functions

BUT

uniform convergence

$$q(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1 & \text{if } x \text{ is rational.} \end{cases}$$

$$q + \frac{7}{\ln(n)} \quad \text{converges uniformly to} \quad q \quad \text{on} \quad \mathbb{R}$$

***despite all of the functions being NOWHERE
continuous on $\mathbb{R}$.***

Dini's Theorem

Dini's Theorem

[Theorem 5.9]

1. f_n converges pointwise to f on $[a, b]$.
2. The f_n are continuous on $[a, b]$.
3. Either

$$f_{n+1} \leq f_n \text{ for all } n \in \mathbb{N}, \text{ or}$$
$$f_{n+1} \geq f_n \text{ for all } n \in \mathbb{N}.$$

$\implies f_n$ converges uniformly to f on $[a, b]$.

Dini's Theorem Example

$$f_n : [4, 11] \rightarrow \mathbb{R} \quad \text{given by} \quad f_n(x) = \frac{x}{3 + x^4/n}.$$

$$\begin{aligned} \frac{x^4}{n+1} < \frac{x^4}{n} &\implies \frac{x}{3 + x^4/(n+1)} > \frac{x}{3 + x^4/n} \\ &\implies f_{n+1}(x) > f_n(x) \end{aligned}$$

Dini's Theorem leads to

f_n converges uniformly to $f(x) = x/3$ on $[4, 11]$.

The
Weierstrass
M-Test

***Weierstrass M-Test** by Example*

$$f_n(x) = \sum_{k=1}^n \frac{\sin^8(e^{k^{293}}x)}{(k+3)^2}.$$

Weierstrass M-Test *by Example*

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For all $x \in \mathbb{R}$ and $k \in \mathbb{N}$

$$\left| \frac{\sin^8(e^{k^{293}x})}{(k+3)^2} \right| < \frac{1}{k^2}.$$

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For all $x \in \mathbb{R}$ and $k \in \mathbb{N}$

$$\left| \frac{\sin^8(e^{k^{293}x})}{(k+3)^2} \right| < \frac{1}{k^2}.$$

*Then using the **Weierstrass M-Test***

$$\sum_{k=1}^n \frac{1}{k^2} \text{ **converges** } \implies f_n \text{ **converges uniformly on } \mathbb{R}.**$$

Position in the Notes

***The end of these slides and the
Week 7 class***

CORRESPOND TO

the end of Section 5.2 on page 70.

The Most Relevant Exercises
(from the subject exercise sets)

1.2., 1.8., 1.16.

