35007

Series

Professor Matt Wand
University of Technology Sydney

First some...

HOUSEKEEPING

First some...

HOUSEKEEPING

The current Hardware Survey results are:

YES	NO
48	0

Series are sequences

Series are sequences

of a specific type.

Series Definition

$a_n, \ n \in \mathbb{N}$, "parent sequence"

gives birth to

a new sequence

$$s_n = \sum_{k=1}^{n} a_k \quad \longleftarrow \text{"series"}$$

Central Series Question

Does the sequence

$$s_n = \sum_{k=1}^{n} a_k$$

converge to a finite number?

Central Series Question

Does the sequence

$$s_n = \sum_{k=1}^{n} a_k \quad \text{converge to a finite number?}$$

Central Series Question

Does the sequence

$$s_n = \sum_{k=1}^{n} a_k \quad \text{converge to a finite number?}$$

The subject notes (and many books) tend to change the k dummy summation variable to n

and write the infinite series as

$$\sum_{n=1}^{\infty} a_n \qquad \left(= \sum_{k=1}^{\infty} a_k = \sum_{\spadesuit=1}^{\infty} a_{\spadesuit} = \lim_{n\to\infty} s_n \right).$$

Some Important Series

Geometric Series

$$\sum_{n=0}^{\infty} r^n$$

Geometric Series

$$\sum_{n=0}^{\infty} r^n$$

converges to $\dfrac{1}{1-r}$ if $|r| < 1$;

Geometric Series

$$\sum_{n=0}^{\infty} r^n$$

converges to $\dfrac{1}{1-r}$ if $|r| < 1$;

and diverges if $|r| \geq 1$.

Reciprocal Power Series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \qquad (\alpha > 0)$$

Reciprocal Power Series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \qquad (\alpha > 0)$$

converges if $\alpha > 1$;

Reciprocal Power Series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \qquad (\alpha > 0)$$

converges if $\alpha > 1$;

and diverges if $0 < \alpha \leq 1$.

Ratio Test for Series

Does $\sum_{n=1}^{\infty} a_n$ converge? $\quad L = \lim_{n \to \infty} \frac{a_{n+1}}{a_n}$

$L < 1 \implies$ YES $\quad L > 1 \implies$ NO

Ratio Test for Series

Does $\sum_{n=1}^{\infty} a_n$ converge? $\quad L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

$L < 1 \implies$ YES $\quad L > 1 \implies$ NO

Class Exercise

Does $\displaystyle\sum_{n=1}^{\infty} \frac{(n+18)^{41}}{3^n}$ converge?

Solution

After letting $a_n = \dfrac{(n+18)^{41}}{3^n}$ we have

$$\frac{a_{n+1}}{a_n} = \frac{(n+19)^{41}}{3^{n+1}} \times \frac{3^n}{(n+18)^{41}}$$

$$= \left(\frac{n+19}{n+18}\right)^{41} \left(\frac{1}{3}\right)$$

$$= \left(\frac{1+19/n}{1+18/n}\right)^{41} \left(\frac{1}{3}\right).$$

Hence

$$L = \left(\frac{1}{3}\right) \left\{\lim_{n\to\infty} \left(\frac{1+19/n}{1+18/n}\right)\right\}^{41} = \left(\frac{1}{3}\right) \times 1 = \frac{1}{3} < 1.$$

By the Ratio Test for Series (Theorem 1.41) $\sum_{n=1}^{\infty} a_n$ converges.

The Limit Comparison Test

Does $\sum_{n=1}^{\infty} a_n$ converge? $\quad \ell = \lim_{n\to\infty} \frac{a_n}{b_n} \quad [a_n, b_n > 0]$

$\ell \neq 0 \implies$ YES if $\sum_{n=1}^{\infty} b_n$ converges

The Limit Comparison Test

Does $\sum_{n=1}^{\infty} a_n$ converge? $\quad \ell = \lim_{n\to\infty} \frac{a_n}{b_n} \quad [a_n, b_n > 0]$

$\ell \neq 0 \implies$ YES if $\sum_{n=1}^{\infty} b_n$ converges

Class Exercise

Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{37n + \cos(n)}{5n^3 + 4^{-n}}$ converges.

Solution

After letting $a_n = \dfrac{5n + \cos(n)}{37n^3 + 4^{-n}}$ and deciding to compare with $b_n = \dfrac{1}{n^2}$ we have

$$\frac{a_n}{b_n} = \frac{37n + \cos(n)}{5n^3 + 4^{-n}} \times \frac{n^2}{1} = \frac{37n^3 + \cos(n)n^2}{5n^3 + 4^{-n}} = \frac{37 + \dfrac{\cos(n)}{n}}{5 + \dfrac{1}{n4^n}}.$$

Hence

$$\ell = \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{37 + \lim_{n\to\infty}\left(\dfrac{\cos(n)}{n}\right)}{5 + \lim_{n\to\infty}\left(\dfrac{1}{n4^n}\right)}.$$

By the Sandwich Theorem (Theorem 1.23) $\lim_{n\to\infty}\left(\dfrac{\cos(n)}{n}\right) = 0$. It is easy to show that $\lim_{n\to\infty}\left(\dfrac{1}{n4^n}\right) = 0$. Therefore $\ell = \dfrac{37}{5} \neq 0$ and by the Limit Comparison Test $\sum_{n=1}^{\infty} a_n$ converges since $\sum_{n=1}^{\infty} 1/n^2$ is an established convergent series.

Series Tests in the Notes

- The Comparison Test
- The Limit Comparison Test
- The Ratio Test
- The nth Root Test
- Lemma 1.36, Theorem 1.47 and Theorem 1.48.

Why Care About Series?

(as junior real analysts)

Series are important and useful for many mathematical problems such as solving differential equations.

A more fundamental role of series is . . . (will come up soon in 35007 Real Analysis)

formal (real analytic) definitions of trigonometric functions.

Formal Definition

For each $x \in \mathbb{R}$ the sine function at x is defined to be

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Further Reading

Section 1.5 of the subjects notes has other aspects about series for you to read about such as

- Alternating series.
- Conditional convergence.
- Exact convergent results

$$\left(\text{e.g. } 1+1/4+1/9+1/16+1/25+\ldots=\frac{\pi^2}{6}\right)$$

Applets Reinforce -ment

One of many useful applet web-sites:

melbapplets.ms.unimelb.edu.au

Of relevance here is the applet named:

Sequences and series

under the Calculus button.

The Most Relevant Exercises
(from the subject exercise sets)

Exercise Set D

♦ ♦ ♦