

35007

Sequences of Integrals

Professor Matt Wand

University of Technology Sydney

Opening Examples

1. $\lim_{n \rightarrow \infty} \int_{\pi}^{2\pi} \frac{\sin^5(n^8 x) + \sqrt{n^2 + 83}}{nx^4 + 1} dx$

2. $\lim_{n \rightarrow \infty} \int_n^{n+5} \frac{1+x}{4+x} dx$

3. $\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \frac{\sqrt{x} e^{-x/n}}{19 + n^4 x^2} dx$

The Limit /
Integral
Interchange
Question

This Topic's Big Question

DOES $\lim_{n \rightarrow \infty} \int_{\pi}^{2\pi} \frac{\sin^5(n^8 x) + \sqrt{n^2 + 83}}{nx^4 + 1} dx$

EQUAL $\int_{\pi}^{2\pi} \lim_{n \rightarrow \infty} \frac{\sin^5(n^8 x) + \sqrt{n^2 + 83}}{nx^4 + 1} dx$

?

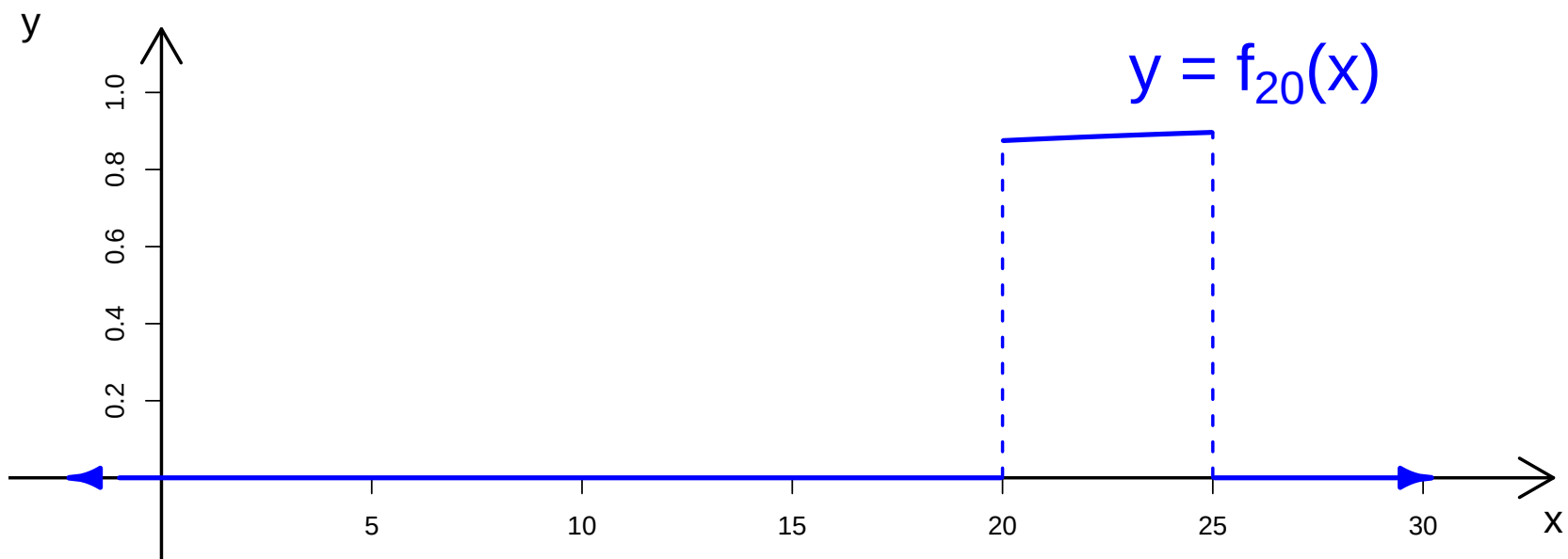
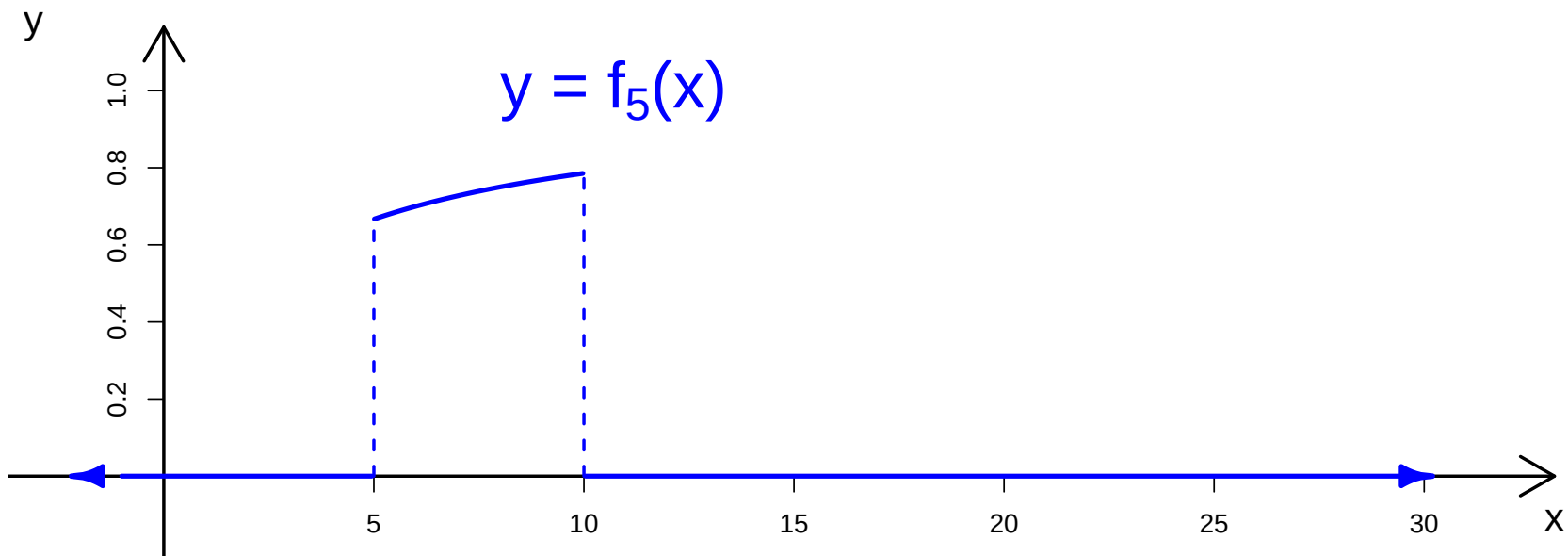
Opening Example 2.

$$\lim_{n \rightarrow \infty} \int_n^{n+5} \frac{1+x}{4+x} dx = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x) dx$$

where

$$f_n(x) = \left(\frac{1+x}{4+x} \right) I(n \leq x \leq n+5)$$

with $I(\mathcal{P}) = 1$ if $\mathcal{P}$ is true
and $I(\mathcal{P}) = 0$ if $\mathcal{P}$ is false [called indicator function notation]



Standard integration steps lead to

$$\int_{-\infty}^{\infty} f_n(x) dx = 5 + 3 \ln \left(\frac{1 + 4/n}{1 + 9/n} \right)$$

$$\implies \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x) dx = 5.$$

Standard integration steps lead to

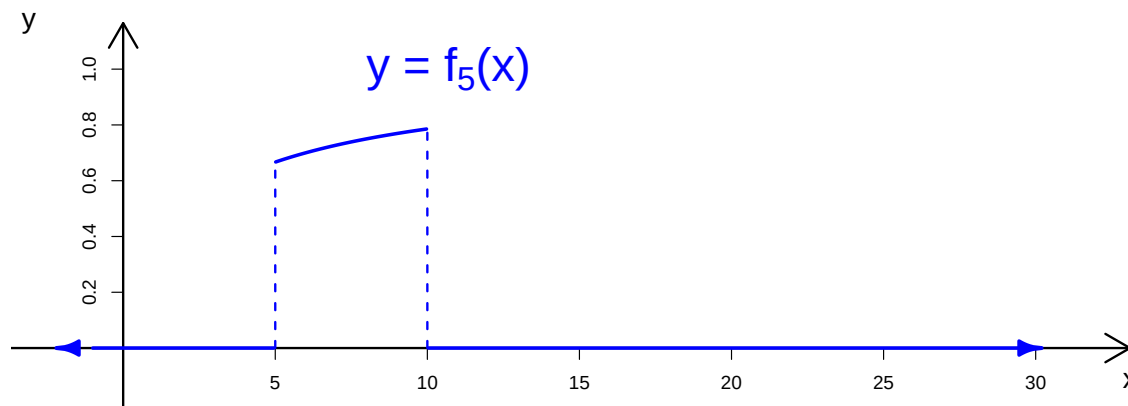
$$\int_{-\infty}^{\infty} f_n(x) dx = 5 + 3 \ln \left(\frac{1 + 4/n}{1 + 9/n} \right)$$

$$\implies \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x) dx = 5.$$

However, for all $x \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} f_n(x) = 0 \implies \int_{-\infty}^{\infty} \lim_{n \rightarrow \infty} f_n(x) dx = 0.$$

Opening Example 2.



$$5 = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x) \neq \int_{-\infty}^{\infty} \lim_{n \rightarrow \infty} f_n(x) dx = 0$$

**so limits and integrals cannot
always be interchanged.**

The Main Limit /

Integral

Interchange

Theorem of 35007

Theorem 5.10

For all $n \in \mathbb{N}$

$f_n : [a, b] \rightarrow \mathbb{R}$ is *Riemann integrable*,

f_n *converges uniformly* to f on $[a, b]$.

$$\implies \lim_{n \rightarrow \infty} \int_a^b f_n(x) = \int_a^b \lim_{n \rightarrow \infty} f_n(x) dx$$

Opening Example 1.

WANTED: $\lim_{n \rightarrow \infty} \int_{\pi}^{2\pi} g_n(x) \, dx$

$$g_n(x) = \frac{\sin^5(n^8 x) + \sqrt{n^2 + 83}}{nx^4 + 1}.$$

Opening Example 1.

WANTED: $\lim_{n \rightarrow \infty} \int_{\pi}^{2\pi} g_n(x) dx$

$$g_n(x) = \frac{\sin^5(n^8 x) + \sqrt{n^2 + 83}}{nx^4 + 1}.$$

Pointwise limit:

$$\lim_{n \rightarrow \infty} g_n(x) = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} \sin^5(n^8 x) + \sqrt{1 + \frac{83}{n^2}}}{x^4 + \frac{1}{n}} = g(x)$$

where $g(x) = \frac{1}{x^4}.$

For all $\pi \leq x \leq 2\pi$, and using *the triangle inequality*

$$\begin{aligned}
 |g_n(x) - g(x)| &= \left| \frac{\frac{1}{n} \sin^5(n^8 x) + \sqrt{1 + \frac{83}{n^2}}}{x^4 + \frac{1}{n}} - \frac{1}{x^4} \right| \\
 &= \left| \frac{\frac{x^4 \sin^5(n^8 x)}{n} + x^4 \sqrt{1 + \frac{83}{n^2}} - x^4 + \frac{1}{n}}{x^4(x^4 + \frac{1}{n})} \right| \\
 &\leq \frac{\left| \frac{x^4 \sin^5(n^8 x)}{n} + x^4 \sqrt{1 + \frac{83}{n^2}} - x^4 + \frac{1}{n} \right|}{x^4(x^4)} \\
 &\leq \frac{|\sin^5(n^8 x)|}{nx^4} + \frac{\sqrt{1 + \frac{83}{n^2}} - 1}{x^4} + \frac{1}{nx^4} \\
 &\leq \frac{1}{nx^4} + \frac{\sqrt{1 + \frac{83}{n^2}} - 1}{x^4} + \frac{1}{nx^4} = \frac{2}{nx^4} + \frac{\left| \sqrt{1 + \frac{83}{n^2}} - 1 \right|}{x^4}.
 \end{aligned}$$

Since

$$x \geq \pi \implies \frac{1}{x^4} \leq \frac{1}{\pi^4}$$

we then have for all $\pi \leq x \leq 2\pi$

$$|g_n(x) - g(x)| \leq \frac{2}{n\pi^4} + \frac{\left| \sqrt{1 + \frac{83}{n^2}} - 1 \right|}{\pi^4} \quad \text{and so}$$

$$0 \leq \sup \left\{ |g_n(x) - g(x)| : \pi \leq x \leq 2\pi \right\} \leq \frac{2}{n\pi^4} + \left(\sqrt{1 + \frac{83}{n^2}} - 1 \right) / \pi^4.$$

Using the **Sandwich Theorem** [Theorem 1.23] we have

$$\lim_{n \rightarrow \infty} \sup \left\{ |g_n(x) - g(x)| : \pi \leq x \leq 2\pi \right\} = 0.$$

From **Theorem S.2** in the supplement

g_n **converges uniformly** to g on $[\pi, 2\pi]$.

The last 2 slides justify

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\pi}^{2\pi} \frac{\sin^5(n^8 x) + \sqrt{n^2 + 83}}{nx^4 + 1} dx \\ &= \int_{\pi}^{2\pi} \lim_{n \rightarrow \infty} \frac{\sin^5(n^8 x) + \sqrt{n^2 + 83}}{nx^4 + 1} dx \\ &= \int_{\pi}^{2\pi} x^{-4} dx = \frac{7}{24\pi^3}. \end{aligned}$$

Simpler Cases

Due to

Dini's Theorem

Dini-Aided Example

WANTED: $\lim_{n \rightarrow \infty} \int_1^9 h_n(x) dx$

where $h_n(x) = \frac{1}{8x + 3 + 17x^6 / \ln(n + 14)}.$

Dini-Aided Example

WANTED: $\lim_{n \rightarrow \infty} \int_1^9 h_n(x) dx$

where $h_n(x) = \frac{1}{8x + 3 + 17x^6 / \ln(n + 14)}.$

Establishing monotonic convergence:

For each $n \in \mathbb{N}$ and $1 \leq x \leq 9$:

$$n + 15 > n + 14 \implies \ln(n + 15) > \ln(n + 14) \implies$$

$$8x + 3 + \frac{17x^6}{\ln(n + 15)} < 8x + 3 + \frac{17x^6}{\ln(n + 14)} \implies h_{n+1}(x) > h_n(x)$$

$$\implies h_n(x) \text{ converges monotonically to } 1/(8x + 3).$$

By Theorem 5.9 (Dini's) the convergence is UNIFORM.

Dini-Justified Interchange

*We are now allowed to interchange
lim and $\int$:*

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_1^9 \frac{1}{8x + 3 + 17x^6 / \ln(n + 14)} dx \\ &= \int_1^9 \lim_{n \rightarrow \infty} \frac{1}{8x + 3 + 17x^6 / \ln(n + 14)} dx \\ &= \int_1^9 \frac{1}{8x + 3} dx = \frac{1}{8} \left[\ln \left(x + \frac{3}{8} \right) \right]_1^9 = \frac{1}{8} \ln \left(\frac{75}{11} \right). \end{aligned}$$

Class Exercise

Determine, with full justification,

$$\lim_{n \rightarrow \infty} \int_2^7 \frac{1}{x^2 + \cos^{12}(x)/n} dx.$$

Class Exercise **SOLUTION**

We want $\lim_{n \rightarrow \infty} \int_2^7 f_n(x) dx$ where $f_n(x) = \frac{1}{x^2 + \cos^{12}(x)/n}$.

For all $n \in \mathbb{N}$ and $x \in [2, 7]$

$$n + 1 > n \quad \implies \quad \cos^{12}(x)/n < \cos^{12}(x)/(n + 1)$$

$$\implies \frac{1}{x^2 + \cos^{12}(x)/n} > \frac{1}{x^2 + \cos^{12}(x)/(n + 1)}$$

$\implies f_n(x)$ converges **monotonically** to $f(x) = \frac{1}{x^2}$ on $[2, 7]$.

By **Dini's Theorem** f_n converges uniformly to f on $[2, 7]$.

By **Theorem 5.10**

$$\lim_{n \rightarrow \infty} \int_2^7 f_n(x) dx = \int_2^7 \lim_{n \rightarrow \infty} f_n(x) dx = \int_2^7 x^{-2} dx = \frac{5}{14}.$$

The Most Relevant Exercises
(from the subject exercise sets)

Exercise Set K

