

UNIVERSITY OF TECHNOLOGY SYDNEY
School of Mathematical and Physical Sciences
35007 Real Analysis

EXERCISE SET L – Solutions

THEME: Integration Methods

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L.1. *Relevant page(s) in the notes: 79*

The integral of interest is

$$I = \int_0^{\infty} \exp(-7x^4) dx.$$

Consider the substitution

$$u = 7x^4 \iff x = 7^{-1/4}u^{1/4}.$$

Then

$$\frac{dx}{du} = \frac{1}{4}7^{-1/4}u^{-3/4} \implies dx = \frac{u^{-3/4} du}{4(7^{1/4})}.$$

Also

$$x = 0 \implies u = 0 \quad \text{and} \quad x = \infty \implies u = \infty.$$

These results lead to

$$I = \frac{1}{4(7^{1/4})} \int_0^{\infty} u^{-3/4} e^{-u} du = \frac{1}{4(7^{1/4})} \int_0^{\infty} u^{1/4-1} e^{-u} du = \frac{\Gamma(\frac{1}{4})}{4(7^{1/4})}.$$

Note that, due to the Gamma function property $\Gamma(z+1) = z\Gamma(z)$, other equivalent answers exist. An example is $I = \Gamma(5/4)/7^{1/4}$.

L.2. *Relevant page(s) in the notes: 79, 80*

Using the definition of the Beta function and equation (6.10) that expresses the Beta function in terms of the Gamma function we have

$$\begin{aligned} \int_0^1 x^{-1/8}(1-x)^{-7/8} dx &= \int_0^1 x^{7/8-1}(1-x)^{1/8-1} dx \\ &= B(7/8, 1/8) \\ &= \frac{\Gamma(7/8)\Gamma(1/8)}{\Gamma(1)} \\ &= \frac{\Gamma(7/8)\Gamma(1/8)}{0!} \\ &= \Gamma(7/8)\Gamma(1/8). \end{aligned}$$

¹This document is less than a few months old which makes errors more likely. Updated versions with later dates may be released during the semester. Notifications will be sent via e-mail whenever an updated version is released.

L.3. Relevant page(s) in the notes: 62

For each $R > 0$ define

$$I(R) \equiv \int_{-R}^R \frac{x}{x^2 + 1} dx.$$

Let

$$u = x^2 + 1 \quad \implies \quad \frac{du}{dx} = 2x \quad \implies \quad du = 2x dx.$$

Also

$$x = -2R \implies u = R^2 + 1 \quad \text{and} \quad x = R \implies u = R^2 + 1.$$

Hence

$$I(R) = \frac{1}{2} \int_{R^2+1}^{R^2+1} u^{-1} du = \frac{1}{2} \left[\ln(u) \right]_{R^2+1}^{R^2+1} = 0.$$

Then, trivially,

$$\lim_{R \rightarrow \infty} I(R) = 0$$

which leads to

$$\text{pv} \int_{-\infty}^{\infty} \frac{x}{x^2 + 1} dx = 0.$$

L.4. Relevant page(s) in the notes: 17, 93

For all $x > 1$ we have

$$\frac{1}{1+x^2} = \frac{1}{x^2} \left\{ \frac{1}{1 - (-1/x^2)} \right\} = \frac{1}{x^2} \sum_{k=0}^{\infty} (-1/x^2)^k.$$

Since $x > 1 \implies |-1/x^2| < 1$, this series converges for all $x > 1$. Therefore,

$$\frac{\ln(x)}{1+x^2} = \frac{\ln(x)}{x^2} \sum_{k=0}^{\infty} (-1/x^2)^k = \sum_{k=0}^{\infty} (-1)^k \ln(x) x^{-2k-2} = \lim_{n \rightarrow \infty} \sum_{k=0}^n (-1)^k \ln(x) x^{-2k-2}.$$

Hence

$$\int_1^{\infty} \frac{\ln(x)}{1+x^2} dx = \int_1^{\infty} \lim_{n \rightarrow \infty} \sum_{k=0}^n (-1)^k \ln(x) x^{-2k-2} dx = \int_1^{\infty} \lim_{n \rightarrow \infty} f_n(x) dx$$

where

$$f_n(x) \equiv \sum_{k=0}^n (-1)^k \ln(x) x^{-2k-2} \quad \text{for all } n \in \mathbb{N} \text{ and } x > 1.$$

From the interchange of integration and limits result in the exercise

$$\begin{aligned} \int_1^{\infty} \lim_{n \rightarrow \infty} f_n(x) dx &= \lim_{n \rightarrow \infty} \int_1^{\infty} f_n(x) dx \\ &= \lim_{n \rightarrow \infty} \int_1^{\infty} \sum_{k=0}^n (-1)^k \ln(x) x^{-2k-2} dx \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^n (-1)^k \int_1^{\infty} \ln(x) x^{-2k-2} dx. \end{aligned}$$

Let

$$u = \ln(x) \implies x = e^u \implies \frac{dx}{du} = e^u \implies dx = e^u du.$$

Noting that

$$x = 1 \implies u = 0 \quad \text{and} \quad x = \infty \implies u = \infty$$

we then have

$$\int_1^\infty \ln(x) x^{-2k-2} dx = \int_0^\infty u e^{-(2k+1)u} du.$$

Using integration by parts

$$\begin{aligned} \int_0^\infty u e^{-(2k+1)u} du &= \left[\frac{u e^{-(2k+1)u}}{\{-(2k+1)\}} \right]_0^\infty + \frac{1}{2k+1} \int_0^\infty e^{-(2k+1)u} du \\ &= -\frac{1}{(2k+1)^2} \left[e^{-(2k+1)u} \right]_0^\infty \\ &= \frac{1}{(2k+1)^2}. \end{aligned}$$

Therefore,

$$\int_1^\infty \frac{\ln(x)}{1+x^2} dx = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{(-1)^k}{(2k+1)^2} = \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^2} = G.$$

L.5. Relevant page(s) in the notes: 17, 69, 70

The goal is to obtain

$$\int_0^1 \frac{x \ln^3(x)}{1-x^4} dx.$$

Consider the substitution

$$x = e^{-u} \implies \frac{dx}{du} = -e^{-u} \implies dx = -e^{-u} du.$$

This substitution is also such that

$$\ln(x) = -u, \quad x = 0 \implies u = \infty \quad \text{and} \quad x = 1 \implies u = 0.$$

We then have

$$\int_0^1 \frac{x \ln^3(x)}{1-x^4} dx = \int_\infty^0 \frac{e^{-u} (-u)^3}{1-e^{-4u}} (-e^{-u}) du = - \int_0^\infty \frac{e^{-2u} u^3}{1-e^{-4u}} du. \quad (1)$$

Since

$$0 < u < \infty \implies -\infty < -4u < 0 \implies 0 < e^{-4u} < 1$$

we have the geometric series result

$$\frac{1}{1-e^{-4u}} = \sum_{k=0}^\infty e^{-4ku} = \lim_{n \rightarrow \infty} \sum_{k=0}^n e^{-4ku}.$$

Substitution of this series into (1) then gives

$$\int_0^1 \frac{x \ln^3(x)}{1-x^4} dx = \int_0^\infty \lim_{n \rightarrow \infty} f_n(u) du$$

where

$$f_n(u) = -e^{-2u} u^3 \sum_{k=0}^n e^{-4ku}.$$

Since

$$f_{n+1}(u) - f_n(u) = -e^{-2u} u^3 \sum_{k=0}^{n+1} e^{-4ku} + e^{-2u} u^3 \sum_{k=0}^n e^{-4ku} = -u^3 e^{-2u-4(n+1)u} \leq 0$$

for all $u \in (0, \infty)$, we have

$$f_{n+1}(u) \leq f_n(u) \quad \text{for all } u \in (0, \infty) \text{ and } n \in \mathbb{N}.$$

So by an extension of Theorem 5.9 to functions on $(0, \infty)$,

$$f_n \text{ converges uniformly to } f \text{ on } (0, \infty).$$

where f is the function such that $\lim_{n \rightarrow \infty} f_n(u) = f(u)$ for each $u > 0$. Therefore, by an extension of Theorem 5.10 to functions on $(0, \infty)$,

$$\int_0^\infty \lim_{n \rightarrow \infty} f_n(u) du = \lim_{n \rightarrow \infty} \int_0^\infty f_n(u) du.$$

Hence

$$\begin{aligned} \int_0^1 \frac{x \ln^3(x)}{1-x^4} dx &= \lim_{n \rightarrow \infty} \int_0^\infty \sum_{k=0}^n (-1) e^{-2u} u^3 e^{-4ku} du \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^n \int_0^\infty (-1) u^3 e^{-(4k+2)u} du. \end{aligned}$$

Now consider the substitution

$$v = (4k+2)u \quad \implies \quad \frac{dv}{du} = (4k+2) \quad \implies \quad dv = (4k+2)du$$

This u to v substitution leads to

$$\begin{aligned}
\int_0^1 \frac{x \ln^3(x)}{1-x^4} dx &= \lim_{n \rightarrow \infty} \sum_{k=0}^n \int_0^\infty (-1) \left(\frac{v}{4k+2} \right)^3 e^{-v} dv / (4k+2) \\
&= \lim_{n \rightarrow \infty} \int_0^\infty (-1) v^{4-1} e^{-v} dv \sum_{k=0}^n \frac{1}{(4k+2)^{2r+1}} \\
&= -\Gamma(4) \sum_{k=0}^\infty \frac{1}{(4k+2)^{2r+1}} \\
&= -\frac{\Gamma(4)}{2^4} \sum_{k=0}^\infty \frac{1}{(2k+1)^4} \\
&= -\frac{3!}{2^4} \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots \right) \\
&= -\frac{6}{16} \left\{ \left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \dots \right) \right. \\
&\quad \left. - \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \dots \right) \right\} \\
&= -\frac{6}{16} \left\{ \left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \dots \right) \right. \\
&\quad \left. - \frac{1}{2^4} \left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots \right) \right\} \\
&= -\frac{6}{16} \times \frac{15}{16} \sum_{k=1}^\infty \frac{1}{k^4} = -\frac{90}{256} \times \frac{\pi^4}{90} = -\frac{\pi^4}{256}.
\end{aligned}$$

In summary,

$$\int_0^1 \frac{x \ln^3(x)}{1-x^4} dx = -\frac{\pi^4}{256}.$$

L.6. Relevant page(s) in the notes: 17

The geometric series result implies that for all $0 < x < 1$ we have

$$\frac{1}{x+1} = \sum_{k=0}^\infty (-x)^k.$$

Therefore,

$$\int_0^1 \frac{\ln^5(x)}{x+1} dx = \int_0^1 \sum_{k=0}^\infty \ln^5(x) (-x)^k dx = \int_0^1 \lim_{n \rightarrow \infty} \sum_{k=0}^n \ln^5(x) (-x)^k dx = \int_0^1 \lim_{n \rightarrow \infty} f_n(x) dx$$

where

$$f_n(x) = \sum_{k=0}^n \ln^5(x) (-x)^k dx.$$

The interchange of integral and limit result (an assumption in the exercise):

$$\int_0^1 \lim_{n \rightarrow \infty} f_n(x) dx = \lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx$$

then leads to

$$\int_0^1 \frac{\ln^5(x)}{x+1} dx = \lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = \lim_{n \rightarrow \infty} \int_0^1 \sum_{k=0}^n \ln^5(x)(-x)^k dx = \lim_{n \rightarrow \infty} \sum_{k=0}^n \int_0^1 \ln^5(x)(-x)^k dx.$$

Now let

$$u = -\ln(x) \implies \frac{du}{dx} = -\frac{1}{x} \implies du = -x^{-1} dx.$$

Therefore,

$$\ln(x) = -u, \quad x = e^{-u} \quad \text{and} \quad dx = -e^{-u} du.$$

Also

$$x = 0 \implies u = \infty \quad \text{and} \quad x = 1 \implies u = 0.$$

Therefore, the substitution from x to u leads to

$$\int_0^1 \frac{\ln^5(x)}{x+1} dx = \lim_{n \rightarrow \infty} \sum_{k=0}^n (-1)^k \int_{\infty}^0 (-u)^5 e^{-ku} (-e^{-u}) du = \lim_{n \rightarrow \infty} \sum_{k=0}^n (-1)^{k+1} \int_0^{\infty} u^5 e^{-(k+1)u} du.$$

Next let

$$v = (k+1)u \implies \frac{dv}{du} = k+1 \implies dv = (k+1)du.$$

Noting that $u = v/(k+1)$ and $du = dv/(k+1)$ we then have

$$\begin{aligned} \int_0^1 \frac{\ln^5(x)}{x+1} dx &= \lim_{n \rightarrow \infty} \sum_{k=0}^n (-1)^{k+1} \int_0^{\infty} \left(\frac{v}{k+1} \right)^5 e^{-v} dv / (k+1) \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{(-1)^{k+1}}{(k+1)^6} \int_0^{\infty} u^{6-1} e^{-u} du \\ &= \Gamma(6)(-1) \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^6}. \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^1 \frac{\ln^5(x)}{x+1} dx &= -5! \left(\frac{1}{1^6} - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \frac{1}{5^6} - \frac{1}{6^6} + \dots \right) \\ &= -120 \left\{ \left(\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \dots \right) \right. \\ &\quad \left. - 2 \left(\frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \frac{1}{8^6} + \frac{1}{10^6} + \dots \right) \right\} \\ &= -120 \left\{ \left(\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \dots \right) \right. \\ &\quad \left. - \frac{2}{2^6} \left(\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \dots \right) \right\} \\ &= -120 \left(1 - \frac{1}{32} \right) \sum_{n=1}^{\infty} \frac{1}{n^6}. \end{aligned}$$

As stated in equation (1.2)

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

so we now have

$$\int_0^1 \frac{\ln^5(x)}{x+1} dx = -\frac{120}{1} \times \frac{31}{32} \times \frac{\pi^6}{945} = -\frac{31\pi^6}{252}.$$

In summary,

$$\int_0^1 \frac{\ln^5(x)}{x+1} dx = -\frac{31\pi^6}{252}.$$

L.7. Relevant page(s) in the notes: 84

For each $t > 0$ define

$$F(t) = \int_0^1 \frac{x^t - 1}{\ln(x)} dx.$$

Then Leibniz's rule (Theorem 6.1) leads to

$$\begin{aligned} F'(t) &= \int_0^1 \frac{\partial}{\partial t} \frac{x^t - 1}{\ln(x)} dx \\ &= \int_0^1 \frac{\partial}{\partial t} \frac{\exp(t \ln(x)) - 1}{\ln(x)} dx \\ &= \int_0^1 \frac{\exp(t \ln(x)) \ln(x)}{\ln(x)} dx \\ &= \int_0^1 \frac{\exp(t \ln(x)) \ln(x)}{\ln(x)} dx \\ &= \int_0^1 x^t dx \\ &= \left[\frac{x^{t+1}}{t+1} \right]_0^1 \\ &= \frac{1}{t+1}. \end{aligned}$$

Hence

$$F(t) = \int \frac{1}{t+1} dt = \ln(t+1) + C \quad \text{for some } C \in \mathbb{R}.$$

Now $F(0) = 0$ so

$$0 = \ln(1) + C = 0 + C \implies C = 0.$$

Thus,

$$F(t) = \ln(t+1).$$

The required integral is

$$F(\sqrt{719\pi^3}) = \ln(\sqrt{719\pi^3} + 1).$$

In summary,

$$\int_0^1 \frac{x^{\sqrt{719\pi^3}} - 1}{\ln(x)} dx = \ln(\sqrt{719\pi^3} + 1).$$

L.8. Relevant page(s) in the notes: 58, 84

For each $t > 0$ define

$$F(t) = \int_0^1 \frac{\ln(1+tx)}{1+x^2} dx.$$

Then the required integral is

$$F(1) = \int_0^1 \frac{\ln(1+x)}{1+x^2} dx. \quad (2)$$

Also note that

$$F(0) = \int_0^1 \frac{\ln(1)}{1+x^2} dx = \int_0^1 0 dx = 0. \quad (3)$$

Using Leibniz's Rule (Theorem 6.1),

$$F'(t) = \int_0^1 \frac{\partial}{\partial t} \frac{\ln(1+tx)}{1+x^2} dx = \int_0^1 \frac{x}{(1+x^2)(1+tx)} dx.$$

To solve this integral we note the following partial fraction decomposition of the integrand:

$$\frac{x}{(1+x^2)(1+tx)} = \frac{1}{1+t^2} \left(\frac{-t}{1+tx} + \frac{2x}{2(1+x^2)} + \frac{t}{1+x^2} \right) \quad (4)$$

The derivation of (4) requires several steps, which are given below in an appendix. From (4), we have

$$\begin{aligned} (1+t^2) \int_0^1 \frac{x}{(1+x^2)(1+tx)} dx &= - \int_0^1 \frac{t}{1+tx} dx + \frac{1}{2} \int_0^1 \frac{2x}{1+x^2} dx + t \int_0^1 \frac{1}{1+x^2} dx \\ &= - \left[\ln(1+tx) \right]_0^1 + \frac{1}{2} \left[\ln(1+x^2) \right]_0^1 + t \left[\tan^{-1}(x) \right]_0^1 \\ &= - \{ \ln(1+t) - \ln(1) \} + \frac{1}{2} \{ \ln(2) - \ln(1) \} + t \left(\frac{\pi}{4} - 0 \right) \\ &= - \ln(1+t) + \frac{1}{2} \ln(2) + \frac{t\pi}{4}. \end{aligned}$$

Therefore,

$$F'(t) = - \frac{\ln(1+t)}{1+t^2} + \frac{\ln(2)}{2} \left(\frac{1}{1+t^2} \right) + \frac{\pi}{8} \left(\frac{2t}{1+t^2} \right).$$

By the Fundamental Theorem of Calculus II (Corollary 4.6),

$$\int_0^1 F'(t) dt = [F(t)]_0^1 = F(1) - F(0).$$

From (3) we have $F(0) = 0$ and so, in view of (2), the quantity of interest is

$$\begin{aligned}
F(1) &= \int_0^1 F'(t) dt \\
&= - \int_0^1 \frac{\ln(1+t)}{1+t^2} dt + \frac{\ln(2)}{2} \int_0^1 \left(\frac{1}{1+t^2} \right) dt + \frac{\pi}{8} \int_0^1 \left(\frac{2t}{1+t^2} \right) dt \\
&= - \int_0^1 \frac{\ln(1+x)}{1+x^2} dx + \frac{\ln(2)}{2} \left[\tan^{-1}(t) \right]_0^1 + \frac{\pi}{8} \left[\ln(1+t^2) \right]_0^1 \\
&= -F(1) + \frac{\ln(2)}{2} \left(\frac{\pi}{4} - 0 \right) + \frac{\pi}{8} \left(\ln(2) - \ln(1) \right) \\
&= -F(1) + \frac{\pi \ln(2)}{4}.
\end{aligned}$$

We then have

$$2F(1) = \frac{\pi \ln(2)}{4} \implies F(1) = \frac{\pi \ln(2)}{8}.$$

Hence, remembering (2),

$$\int_0^1 \frac{\ln(1+x)}{1+x^2} dx = \frac{\pi \ln(2)}{8}.$$

Appendix: Derivation of (4)

We start by hoping that there is an expression of the form

$$\frac{x}{(1+x^2)(1+tx)} = \frac{A_t + B_t x}{1+x^2} + \frac{C_t + D_t x}{1+tx} \quad (5)$$

for quantities A_t, B_t, C_t and D_t that can depend on t (but not on x). Multiplication of both sides of (5) leads to

$$\begin{aligned}
x &= (A_t + B_t x)(1+tx) + (C_t + D_t x)(1+x^2) \\
&= A_t + A_t tx + B_t x + B_t tx^2 + C_t + C_t x^2 + D_t x + D_t x^3 \\
&= (A_t + C_t) + (A_t t + B_t + D_t)x + (B_t t + C_t)x^2 + (D_t)x^3.
\end{aligned}$$

Therefore, we have

$$(0) + (1)x + (0)x^2 + (0)x^3 = (A_t + C_t) + (A_t t + B_t + D_t)x + (B_t t + C_t)x^2 + (D_t)x^3.$$

Matching the coefficients of each power of the cubic polynomials on each side of this expression leads to

$$A_t + C_t = 0, \quad A_t t + B_t + D_t = 1, \quad B_t t + C_t = 0 \quad \text{and} \quad D_t = 0.$$

The first equality gives $C_t = -A_t$ and substitution of this and $D_t = 0$ into the middle two equalities gives

$$A_t t + B_t = 1 \quad \text{and} \quad B_t t - A_t = 0.$$

The second of these equalities gives $A_t = tB_t$, and so substitution into the first equality gives

$$(t^2 + 1)B_t = 1 \implies B_t = 1/(t^2 + 1).$$

This leads to the explicit expressions

$$A_t = t/(t^2 + 1) \quad \text{and} \quad C_t = -t/(t^2 + 1).$$

Substitution of these results into (5) leads to

$$\begin{aligned} \frac{x}{(1+x^2)(1+tx)} &= \frac{(t+x)/(t^2+1)}{1+x^2} - \frac{t/(t^2+1)}{1+tx} \\ &= \frac{1}{1+t^2} \left(\frac{t+x}{1+x^2} - \frac{t}{1+tx} \right) \\ &= \frac{1}{1+t^2} \left(-\frac{t}{1+tx} + \frac{2x}{2(1+x^2)} + \frac{t}{1+x^2} \right) \end{aligned}$$

which is equation (4).

L.9. Relevant page(s) in the notes: 84

In the notation of Theorem 6.1

$$u_1(t) = t^2, \quad u_2(t) = 2 \cosh(t) \quad \text{and} \quad f(x, t) = e^{-x(t^4+8)}.$$

Then

$$\frac{\partial}{\partial t} e^{-x(t^4+8)} = -4t^3 x e^{-x(t^4+8)}, \quad u_1'(t) = 2t \quad \text{and} \quad u_2'(t) = 2 \sinh(t).$$

Hence, from Leibniz's rule (Theorem 6.1),

$$F'(t) = -4t^3 \int_{t^2}^{2 \cosh(t)} x e^{-x(t^4+8)} dx + 2 \sinh(t) e^{-2 \cosh(t)(t^4+8)} - 2t e^{-t^2(t^4+8)}.$$

L.10. Relevant page(s) in the notes: 84, 87, 88

For each $t \in \mathbb{R}$ define

$$F(t) = \int_{-\infty}^{\infty} \frac{\cos(tx)}{x^2 + 1} dx.$$

The fact that $\cos(tx)/(x^2 + 1)$ is an even function implies

$$F(t) = 2 \int_0^{\infty} \frac{\cos(tx)}{x^2 + 1} dx.$$

Leibniz's rule (Theorem 6.1) leads to

$$\begin{aligned}
F'(t) &= 2 \int_0^\infty \frac{\partial}{\partial t} \frac{\cos(tx)}{1+x^2} dx \\
&= -2 \int_0^\infty \frac{x \sin(tx)}{1+x^2} dx \\
&= -2 \int_0^\infty \frac{x^2 \sin(tx)}{x(1+x^2)} dx \\
&= -2 \int_0^\infty \frac{(x^2+1-1) \sin(tx)}{x(1+x^2)} dx \\
&= -2 \int_0^\infty \frac{\sin(tx)}{x} dx + \int_0^\infty \frac{\sin(tx)}{x(1+x^2)} dx \\
&= -2 \int_0^\infty \frac{\sin(tx)}{tx} t dx + 2 \int_0^\infty \frac{\sin(tx)}{x(1+x^2)} dx.
\end{aligned}$$

For the first integral make the substitution

$$u = tx \quad \implies \quad \frac{du}{dx} = t \quad \implies \quad du = t dx.$$

This leads to

$$F'(t) = -2 \int_0^\infty \frac{\sin(u)}{u} du + 2 \int_0^\infty \frac{\sin(tx)}{x(1+x^2)} dx = -\pi + 2 \int_0^\infty \frac{\sin(tx)}{x(1+x^2)} dx$$

which uses the integral result

$$\int_0^\infty \frac{\sin(x)}{x} dx = \frac{\pi}{2}$$

that is proven (also via Leibniz's rule) in Example 6.6. Another application of Leibniz's rule to

$$F'(t) = -\pi + 2 \int_0^\infty \frac{\sin(tx)}{x(1+x^2)} dx$$

leads to

$$F''(t) = 2 \int_0^\infty \frac{\partial}{\partial t} \frac{\sin(tx)}{x(1+x^2)} dx = 2 \int_0^\infty \frac{\cos(tx)}{1+x^2} dx = F(t) \quad \implies \quad F(t) = F''(t).$$

Now note that

$$F(t) = F''(t)$$

corresponds to the simple differential equation

$$y = \frac{d^2 y}{dt^2} \quad \text{that has solutions} \quad y = C_1 e^t + C_2 e^{-t}.$$

Therefore

$$F(t) = C_1 e^t + C_2 e^{-t} \quad \text{for some} \quad C_1, C_2 \in \mathbb{R}. \quad (6)$$

It follows from this that

$$F'(t) = C_1 e^t - C_2 e^{-t}. \quad (7)$$

Now note that

$$\begin{aligned} F(0) &= \int_{-\infty}^{\infty} \frac{\cos(0)}{x^2 + 1} dx = \int_{-\infty}^{\infty} \frac{1}{x^2 + 1} dx = \left[\tan^{-1}(x) \right]_{-\infty}^{\infty} \\ &= \tan^{-1}(\infty) - \tan^{-1}(-\infty) = \frac{\pi}{2} - \left(-\frac{\pi}{2} \right) = \pi. \end{aligned}$$

Plugging this into (6) leads to

$$C_1 + C_2 = \pi. \quad (8)$$

Also, from the above expression for $F'(t)$,

$$F'(0) = -\pi + 2 \int_0^{\infty} \frac{\sin(0)}{x(1+x^2)} dx = -\pi + 0 = -\pi.$$

Plugging this into (7) leads to

$$C_1 - C_2 = -\pi \quad (9)$$

Adding equations (8) and (9) gives $C_1 = 0$. Quickly after this we have

$$C_2 = \pi \implies F(t) = \pi e^{-t}.$$

Therefore

$$\int_{-\infty}^{\infty} \frac{\cos(tx)}{x^2 + 1} dx = F(1) = \frac{\pi}{e}.$$

In summary

$$\int_{-\infty}^{\infty} \frac{\cos(tx)}{x^2 + 1} dx = \frac{\pi}{e}.$$

