

UNIVERSITY OF TECHNOLOGY SYDNEY
School of Mathematical and Physical Sciences
35007 Real Analysis

EXERCISE SET K – Solutions

THEME: Sequences of Integrals

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K.1. *Relevant page(s) in the notes: 69, 70*

For each $n \in \mathbb{N}$ define $f_n : [0, M] \rightarrow \mathbb{R}$ by

$$f_n(x) = e^{-n(x^4+1)}.$$

Then for each $x \in [0, M]$ and $n \in \mathbb{N}$,

$$n+1 > n \implies (n+1)(x^4+1) > n(x^4+1) \implies e^{-(n+1)(x^4+1)} < e^{-n(x^4+1)}$$

which implies that

$$f_{n+1}(x) < f_n(x) \quad \text{for all } x \in [0, M] \text{ and } n \in \mathbb{N}.$$

In addition,

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} e^{-n(x^4+1)} = 0 \quad \text{for all } x \in [0, M].$$

Therefore f_n converges monotonically to

$$f(x) = 0 \quad \text{on } x \in [0, M].$$

and so by Theorem 5.9 f_n converges to f uniformly on $[0, M]$. Therefore, by Theorem 5.10,

$$\lim_{n \rightarrow \infty} \int_0^M f_n(x) dx = \int_0^M f(x) dx = \int_0^M 0 dx = 0.$$

K.2. *Relevant page(s) in the notes: 12, 70, Supplementary Results*

Let

$$f_n(x) = \frac{(-1)^n \sin^{293}(n^7 \tanh(x))}{\ln(\ln(n+4))}.$$

Then, since $|\sin(t)| \leq 1$ for all $t \in \mathbb{R}$,

$$|\sin(n^7 \tanh(x))| \leq 1 \implies |\sin(n^7 \tanh(x))|^{293} \leq 1$$

$$\implies \frac{|\sin^{293}(n^7 \tanh(x))|}{\ln(\ln(n+4))} \leq \frac{1}{\ln(\ln(n+4))}$$

$$\implies |f_n(x)| \leq \frac{1}{\ln(\ln(n+4))} \quad \text{for all } x \in \mathbb{R}.$$

¹This document is less than a few months old which makes errors more likely. Updated versions with later dates may be released during the semester. Notifications will be sent via e-mail whenever an updated version is released.

Therefore

$$a_n \leq \sup_{x \in [-37, 88]} |f_n(x) - f(x)| \leq \frac{1}{\ln(\ln(n+4))} \quad \text{for all } n \in \mathbb{N}$$

where

$$a_n = 0 \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad f(x) = 0 \quad \text{for all } x \in [-37, 88].$$

Since

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{\ln(\ln(n+4))} = 0$$

by the Sandwich Theorem (Theorem 1.23)

$$\lim_{n \rightarrow \infty} \sup_{x \in [-37, 88]} |f_n(x) - f(x)| = 0.$$

Therefore, by Theorem S.2 (Supplementary Results)

$$f_n \text{ converges uniformly to } f \text{ on } [-37, 88].$$

So, from Theorem 5.10,

$$\lim_{n \rightarrow \infty} \int_{-37}^{88} f_n(x) dx = \int_{-37}^{88} f(x) dx = \int_{-37}^{88} 0 dx = 0.$$

In summary,

$$\lim_{n \rightarrow \infty} \int_{-37}^{88} \frac{(-1)^n \sin^{293}(n^7 \tanh(x))}{\ln(\ln(n+4))} dx = 0.$$

K.3. Relevant page(s) in the notes: 69, 70

For each $n \in \mathbb{N}$ define $f_n : [1, 7] \rightarrow \mathbb{R}$ by

$$f_n(x) \equiv \frac{n}{3nx^2 + 5n + 1187} = \frac{1}{3x^2 + 5 + 1187/n}.$$

Then for each $x \in [1, 7]$ and $n \in \mathbb{N}$,

$$n+1 > n \implies 1187/(n+1) < 1187/n \implies 3x^2 + 5 + 1187/(n+1) < 3x^2 + 5 + 1187/n$$

which implies that

$$f_{n+1}(x) < f_n(x) \quad \text{for all } x \in [1, 7] \text{ and } n \in \mathbb{N}.$$

In addition,

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{1}{3x^2 + 5 + 1187/n} = \frac{1}{3x^2 + 5} \quad \text{for all } x \in [1, 7].$$

Therefore f_n converges monotonically to

$$f(x) = \frac{1}{3x^2 + 5} \quad \text{on } x \in [1, 7].$$

and so by Theorem 5.9 f_n converges to f uniformly on $[1, 7]$. Therefore, by Theorem 5.10,

$$\lim_{n \rightarrow \infty} \int_1^7 f_n(x) dx = \int_1^7 f(x) dx = \int_1^7 \frac{1}{3x^2 + 5} dx = \frac{1}{5} \int_1^7 \frac{1}{(\sqrt{3/5}x)^2 + 1} dx.$$

To evaluate the last-written integral make the substitution

$$u = \sqrt{3/5}x \implies x = \sqrt{5/3}u \implies \frac{dx}{du} = \sqrt{5/3} \implies dx = \sqrt{5/3} du.$$

Also

$$x = 1 \implies u = \sqrt{3/5} \quad \text{and} \quad x = 7 \implies u = 7\sqrt{3/5}.$$

This leads to

$$\lim_{n \rightarrow \infty} \int_1^7 f_n(x) dx = \frac{\sqrt{3}}{5\sqrt{5}} \int_{\sqrt{3/5}}^{7\sqrt{3/5}} \frac{1}{u^2 + 1} du = \frac{\sqrt{5}}{5\sqrt{3}} \left[\tan^{-1}(u) \right]_{\sqrt{3/5}}^{7\sqrt{3/5}}.$$

Straightforward algebraic steps then lead to the following fully simplified answer to the exercise:

$$\frac{\sqrt{15}}{15} \left\{ \tan^{-1} \left(\frac{7\sqrt{15}}{5} \right) - \tan^{-1} \left(\frac{\sqrt{15}}{5} \right) \right\}.$$

K.4. Relevant page(s) in the notes: 6, 12, 70, Supplementary Results

For all $x \in \mathbb{R}$ and $n \in \mathbb{N}$ let

$$f_n(x) = \frac{(n+9) \cosh(x) + \cos(n!x^{43})}{5n+61} \quad \text{and} \quad f(x) = \frac{\cosh(x)}{5}.$$

Then, with use of the triangle inequality and noting that $|\cos(t)| \leq 1$ for all $t \in \mathbb{R}$,

$$\begin{aligned} |f_n(x) - f(x)| &= \left| \frac{(n+9) \cosh(x) + \cos(n!x^{43})}{5n+61} - \frac{\cosh(x)}{5} \right| \\ &\leq \cosh(x) \left| \frac{n+9}{5n+61} - \frac{1}{5} \right| + \frac{|\cos(n!x^{43})|}{5n+61} \\ &\leq \cosh(x) \left| \frac{n+9}{5n+61} - \frac{1}{5} \right| + \frac{1}{5n+61} \\ &= \cosh(x) \left\{ \frac{16}{5(5n+61)} \right\} + \frac{1}{5n+61}. \end{aligned}$$

Noting that the cosh function is U-shaped and symmetric about $x = 0$ it is easy to show that

$$\sup\{\cosh(x) : -14 \leq x \leq 8\} = \cosh(14).$$

Hence

$$\sup\{|f_n(x) - f(x)| : -14 \leq x \leq 8\} \leq \cosh(14) \left\{ \frac{16}{5(5n+61)} \right\} + \frac{1}{5n+61}.$$

Therefore,

$$a_n \leq \sup\{|f_n(x) - f(x)| : -14 \leq x \leq 8\} \leq \cosh(14) \left\{ \frac{16}{5(5n+61)} \right\} + \frac{1}{5n+61} \quad \text{for all } n \in \mathbb{N},$$

where $a_n = 0$ for all $n \in \mathbb{N}$. Since

$$\lim_{n \rightarrow \infty} a_n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \left[\cosh(14) \left\{ \frac{16}{5(5n+61)} \right\} + \frac{1}{5n+61} \right] = 0$$

we have by the Sandwich Theorem (Theorem 1.23)

$$\lim_{n \rightarrow \infty} \sup\{|f_n(x) - f(x)| : -14 \leq x \leq 8\} = 0$$

and by Theorem S.2 (Supplementary Results)

$$f_n \text{ converges uniformly to } f \text{ on } [-14, 8].$$

Therefore, by Theorem 5.10, and noting the derivative result $\sinh'(x) = \cosh(x)$ we have

$$\begin{aligned} \frac{1}{5} \int_{-14}^8 \cosh(x) dx &= \frac{1}{5} \left[\sinh(x) \right]_{-14}^8 \\ &= \frac{\sinh(8) - \sinh(-14)}{5} \\ &= \frac{\sinh(8) + \sinh(14)}{5} \\ &= \frac{e^8 - e^{-8} + e^{14} - e^{-14}}{10} \\ &= \frac{e^{28} + e^{22} - e^6 - 1}{10e^{14}}. \end{aligned}$$

In summary,

$$\lim_{n \rightarrow \infty} \int_{-14}^8 \frac{(n+9) \cosh(x) + \cos(n!x^{43})}{5n+61} dx = \frac{e^{28} + e^{22} - e^6 - 1}{10e^{14}}.$$

K.5. Relevant page(s) in the notes: 69, 70

For each $n \in \mathbb{N}$ define $f_n : [2, 3] \rightarrow \mathbb{R}$ by

$$f_n(x) \equiv \frac{n}{7nx^4 + 41} = \frac{1}{7x^4 + 41/n}.$$

Then for each $x \in [2, 3]$ and $n \in \mathbb{N}$,

$$n+1 > n \implies 41/(n+1) < 41/n \implies 5x^4 + 41/(n+1) < 5x^4 + 41/n$$

which implies that

$$f_{n+1}(x) < f_n(x) \quad \text{for all } x \in [2, 3] \text{ and } n \in \mathbb{N}.$$

In addition,

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{1}{7x^4 + 41/n} = \frac{1}{7x^4} \quad \text{for all } x \in [2, 3].$$

Therefore f_n converges monotonically to

$$f(x) = \frac{1}{7x^4} \quad \text{on } x \in [2, 3].$$

and so by Theorem 5.9 f_n converges to f uniformly on $[2, 3]$. Therefore, by Theorem 5.10,

$$\lim_{n \rightarrow \infty} \int_2^3 f_n(x) dx = \int_2^3 f(x) dx = \frac{1}{7} \int_2^3 x^{-4} dx = \frac{1}{7} \left[\frac{x^{-3}}{-3} \right]_2^3 = \frac{1}{21} (2^{-3} - 3^{-3}) = \frac{19}{4536}.$$

In summary,

$$\lim_{n \rightarrow \infty} \int_2^3 \frac{n}{7nx^4 + 41} dx = \frac{19}{4536}.$$

K.6. Relevant page(s) in the notes: Supplementary Results

For the left-hand side first note that

$$\int_0^1 f_n(x) dx = \int_0^1 nx^{n-1} dx = [x^n]_0^1 = 1^n - 0^n = 1 \quad \text{for all } n \in \mathbb{N}.$$

Therefore, trivially,

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = \lim_{n \rightarrow \infty} 1 = 1.$$

For the right-hand side note that, for each $0 < x < 1$,

$$\left| \frac{f_{n+1}(x)}{f_n(x)} \right| = \left| \frac{(n+1)x^n}{nx^{n-1}} \right| = (1 + 1/n)x.$$

Hence

$$\lim_{n \rightarrow \infty} \left| \frac{f_{n+1}(x)}{f_n(x)} \right| = x < 1.$$

So by the Ratio Test for Sequences (Theorem S.1 of the Supplementary Results)

$$\lim_{n \rightarrow \infty} f_n(x) = 0 \quad \text{for all } 0 < x < 1.$$

It follows that

$$\int_0^1 \lim_{n \rightarrow \infty} f_n(x) dx = \int_0^1 0 dx = 0.$$

In summary,

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = 0.$$

Thus,

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx \neq \int_0^1 \lim_{n \rightarrow \infty} f_n(x) dx.$$

K.7. Relevant page(s) in the notes: Supplementary Results

(a) For each $n \in \mathbb{N}$ and $x = 0$ we have $f_n(x) = 0$ so, trivially,

$$\lim_{n \rightarrow \infty} f_n(x) = 0 = f(x) \quad \text{for } x = 0. \quad (1)$$

For each $n \in \mathbb{N}$ and $x \neq 0$ we have

$$\lim_{n \rightarrow \infty} \left| \frac{f_{n+1}(x)}{f_n(x)} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)xe^{-(n+1)x^2}}{nxe^{-nx^2}} = e^{-x^2} \lim_{n \rightarrow \infty} (1 + 1/n) = e^{-x^2} < 1.$$

By the Ratio Test for Sequences (Theorem S.1 of the Supplementary Results)

$$\lim_{n \rightarrow \infty} f_n(x) = 0 = f(x) \quad \text{for all } x \neq 0. \quad (2)$$

Combining (1) and (2) we have

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \quad \text{for all } x \in \mathbb{R}$$

so f_n converges to f pointwise over $\mathbb{R}$.

(b) Note that

$$|f_n(x) - f(x)| = n|x|e^{-nx^2} \quad \text{which is an even function.}$$

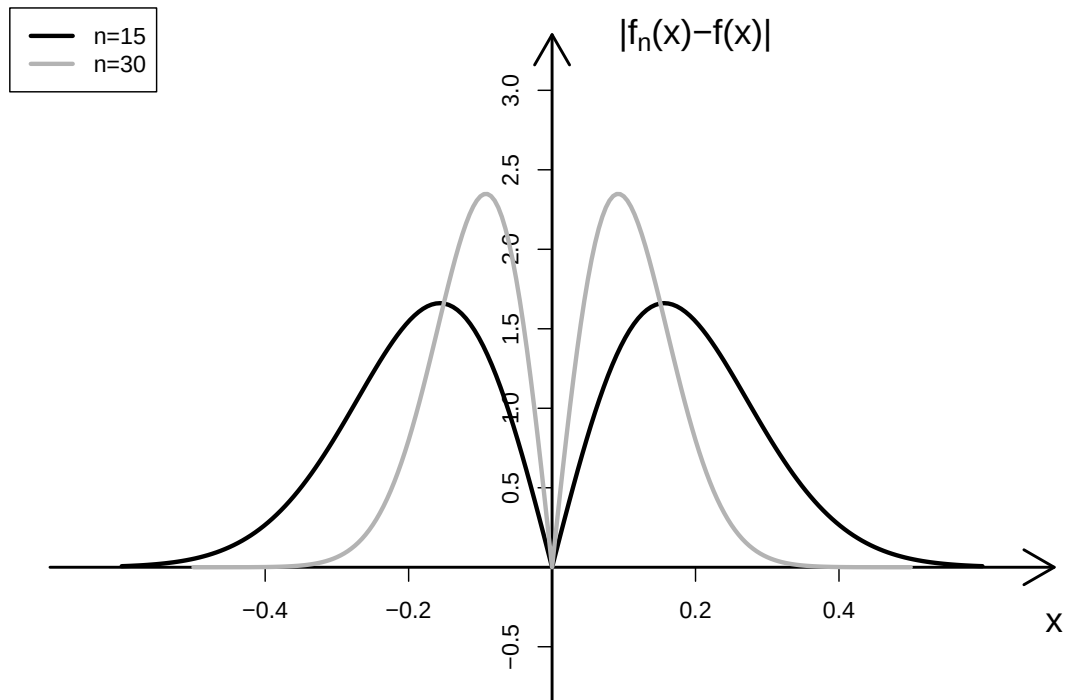
For $x \geq 0$

$$|f_n(x) - f(x)| = nxe^{-nx^2} \implies \frac{d}{dx}|f_n(x) - f(x)| = ne^{-nx^2}(1 - 2nx^2)$$

which leads to $1/\sqrt{2n}$ being the only stationary value of $|f_n(x) - f(x)|$ for $x \in [0, \infty)$. Also,

$$\frac{d}{dx}|f_n(x) - f(x)| > 0 \quad \text{for } 0 < x < \frac{1}{\sqrt{2n}} \quad \text{and} \quad \frac{d}{dx}|f_n(x) - f(x)| < 0 \quad \text{for } x > \frac{1}{\sqrt{2n}}$$

which implies that $|f_n(x) - f(x)|$ achieves its maximum value over $[0, \infty)$ at $x = 1/\sqrt{2n}$. Since $|f_n(x) - f(x)|$ is an even function it achieves its maximum value over $(-\infty, 0]$ at $x = -1/\sqrt{2n}$. The accompanying figure plots $|f_n(x) - f(x)|$ for $n = 15$ and $n = 30$.



Hence,

$$\sup\{|f_n(x) - f(x)| : x \in \mathbb{R}\} = \left|f_n(1/\sqrt{2n}) - f(1/\sqrt{2n})\right| = \sqrt{n/(2e)}.$$

Therefore, the sequence $\sup\{|f_n(x) - f(x)| : x \in \mathbb{R}\}$ diverges and by Theorem S.2 in the Supplementary Results f_n does not converge uniformly to f on $\mathbb{R}$.

(c) Since $\lim_{n \rightarrow \infty} f_n(x) = f(x) = 0$ for all $x \in \mathbb{R}$ we have

$$\int_{-\infty}^{\infty} \lim_{n \rightarrow \infty} f_n(x) dx = \int_{-\infty}^{\infty} 0 dx = 0.$$

Next note that for each $n \in \mathbb{N}$,

$$\int_{-\infty}^{\infty} f_n(x) dx = \int_{-\infty}^{\infty} nxe^{-nx^2} dx.$$

Let

$$u = nx^2 \quad \Longrightarrow \quad \frac{du}{dx} = 2nx \quad \Longrightarrow \quad du = 2nx dx.$$

Also,

$$x = -\infty \quad \Longrightarrow \quad u = 0 \quad \text{and} \quad x = \infty \quad \Longrightarrow \quad u = \infty$$

which leads to

$$\int_{-\infty}^{\infty} f_n(x) dx = \frac{1}{2} \int_0^{\infty} e^{-u} du = \frac{1}{2} \left[-e^{-u} \right]_0^{\infty} = \frac{1}{2} \quad \text{for all } n \in \mathbb{N}.$$

Therefore,

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x) dx = \frac{1}{2}.$$

To summarise,

$$0 = \int_{-\infty}^{\infty} \lim_{n \rightarrow \infty} f_n(x) dx \neq \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x) dx = \frac{1}{2}.$$

K.8. Relevant page(s) in the notes: 69, 70

For each $n \in \mathbb{N}$ define $f_n : [0, 1] \rightarrow \mathbb{R}$ by

$$f_n(x) = \frac{x^2}{1 + e^{-nx}}.$$

Then for each $x \in [0, 1]$ and $n \in \mathbb{N}$,

$$n+1 > n \quad \Longrightarrow \quad -(n+1)x \leq -nx \quad \Longrightarrow \quad 1 + e^{-(n+1)x} \leq 1 + e^{-nx} \quad \Longrightarrow \quad \frac{x^2}{1 + e^{-(n+1)x}} \geq \frac{x^2}{1 + e^{-nx}}$$

which implies that

$$f_{n+1}(x) \geq f_n(x) \quad \text{for all } x \in [0, 1] \text{ and } n \in \mathbb{N}.$$

In addition,

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{x^2}{1 + e^{-nx}} = x^2 \quad \text{for all } x \in [0, 1].$$

Therefore f_n converges monotonically to

$$f(x) = x^2 \quad \text{on } x \in [0, 1].$$

and so by Theorem 5.9 f_n converges to f uniformly on $[0, 1]$. Therefore, by Theorem 5.10,

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = \int_0^1 f(x) dx = \int_0^1 x^2 dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3}.$$

K.9. Relevant page(s) in the notes: 6, 12, 70, Supplementary Results

For all $x \in \mathbb{R}$ and $n \in \mathbb{N}$ let

$$f_n(x) = \left(1 + \frac{(-1)^n}{n} \right) (x - e^{-n})^2 \quad \text{and} \quad f(x) = x^2.$$

Then, using the triangle inequality,

$$\begin{aligned} |f_n(x) - f(x)| &= \left| \left(1 + \frac{(-1)^n}{n} \right) (x - e^{-n})^2 - x^2 \right| \\ &= \left| \left(1 + \frac{(-1)^n}{n} \right) (x^2 - 2xe^{-n} + e^{-2n}) - x^2 \right| \\ &= \left| \left(1 + \frac{(-1)^n}{n} \right) (x^2 - 2xe^{-n} + e^{-2n}) - x^2 \right| \\ &= \left| -2xe^{-n} + e^{-2n} + \frac{(-1)^n}{n} x^2 - \frac{2(-1)^n e^{-n}}{n} x + \frac{(-1)^n e^{-2n}}{n} \right| \\ &\leq 2e^{-n}|x| + e^{-2n} + (1/n)x^2 + (2/n)|x| + \frac{e^{-2n}}{n} \\ &= e^{-2n}(1 + 1/n) + 2(e^{-n} + 1/n)|x| + (1/n)x^2. \end{aligned}$$

Therefore, for all $n \in \mathbb{N}$,

$$\begin{aligned} \sup\{|f_n(x) - f(x)| : -3 \leq x \leq 11\} &\leq e^{-2n}(1 + 1/n) + 2(e^{-n} + 1/n) \sup\{|x| : -3 \leq x \leq 11\} \\ &\quad + (1/n) \sup\{x^2 : -3 \leq x \leq 11\} \\ &= e^{-2n}(1 + 1/n) + 2(e^{-n} + 1/n)11 + (121/n). \end{aligned}$$

It follows immediately that

$$b_n \leq \sup\{|f_n(x) - f(x)| : -3 \leq x \leq 11\} \leq e^{-2n}(1 + 1/n) + 2(e^{-n} + 1/n)11 + (121/n).$$

where $b_n = 0$ for all $n \in \mathbb{N}$. Since

$$\lim_{n \rightarrow \infty} b_n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \{e^{-2n}(1 + 1/n) + 2(e^{-n} + 1/n)11 + (121/n)\} = 0,$$

from the Sandwich Theorem (Theorem 1.23)

$$\lim_{n \rightarrow \infty} \sup\{|f_n(x) - f(x)| : -3 \leq x \leq 11\} = 0$$

and by Theorem S.2 (Supplementary Results)

f_n converges uniformly to f on $[-3, 11]$.

Therefore, by Theorem 5.10

$$\begin{aligned}\lim_{n \rightarrow \infty} \int_{-3}^{11} f_n(x) dx &= \int_{-3}^{11} \lim_{n \rightarrow \infty} f_n(x) dx = \int_{-3}^{11} f(x) dx \\ &= \int_{-3}^{11} x^2 dx \\ &= \left[\frac{x^3}{3} \right]_{-3}^{11} \\ &= \frac{11^3 - (-3)^3}{3} \\ &= \frac{1358}{3}.\end{aligned}$$

In summary,

$$\lim_{n \rightarrow \infty} \int_{-3}^{11} \left(1 + \frac{(-1)^n}{n} \right) (x - e^{-n})^2 dx = \frac{1358}{3}.$$

K.10. Relevant page(s) in the notes: 12, 70, Supplementary Results

For all $x \in \mathbb{R}$ and $n \in \mathbb{N}$ let

$$f_n(x) = \frac{e^x}{\left(1 + e^x + \frac{1}{4\sqrt{n}}\right)^2} \quad \text{and} \quad f(x) = \frac{e^x}{(1 + e^x)^2}.$$

Then

$$\begin{aligned}|f_n(x) - f(x)| &= e^x \left| \frac{1}{\left(1 + e^x + \frac{1}{4\sqrt{n}}\right)^2} - \frac{1}{(1 + e^x)^2} \right| \\ &= e^x \left| \frac{\left(1 + e^x + \frac{1}{4\sqrt{n}}\right)^2 - (1 + e^x)^2}{\left(1 + e^x + \frac{1}{4\sqrt{n}}\right)^2 (1 + e^x)^2} \right| \\ &= \frac{1}{4\sqrt{n}} \left\{ \frac{e^x(1 + 2e^x + \frac{1}{4\sqrt{n}})}{\left(1 + e^x + \frac{1}{4\sqrt{n}}\right)^2 (1 + e^x)^2} \right\}.\end{aligned}$$

Now note that

$$0 < \frac{1}{4\sqrt{n}} < 1 \quad \text{for all } n \in \mathbb{N}. \quad (3)$$

Then (3) implies that

$$1 + 2e^x + \frac{1}{4\sqrt{n}} < 1 + 2e^x + 1 = 2(e^x + 1) \quad (4)$$

as well as

$$1 + e^x + \frac{1}{4\sqrt{n}} > 1 + e^x \quad \implies \quad \frac{1}{1 + e^x + \frac{1}{4\sqrt{n}}} < \frac{1}{1 + e^x} < 1. \quad (5)$$

Application of inequalities (4) and (5) into the above expression for $|f_n(x) - f(x)|$ leads to the inequalities

$$|f_n(x) - f(x)| < \frac{e^x(e^x + 1)}{2\sqrt{n}(1 + e^x)^2} = \frac{1}{2\sqrt{n}(1 + e^{-x})} < \frac{1}{2\sqrt{n}} \quad \text{for all } x \in \mathbb{R} \text{ and } n \in \mathbb{N}.$$

Therefore,

$$a_n \leq \sup\{|f_n(x) - f(x)| : -1 \leq x \leq 2\} \leq \frac{2}{\sqrt{n}} \quad \text{for all } n \in \mathbb{N},$$

where $a_n = 0$ for all $n \in \mathbb{N}$. Since

$$\lim_{n \rightarrow \infty} a_n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{2}{\sqrt{n}} = 0$$

we have by the Sandwich Theorem (Theorem 1.23)

$$\lim_{n \rightarrow \infty} \sup\{|f_n(x) - f(x)| : -1 \leq x \leq 2\} = 0$$

and by Theorem S.2 (Supplementary Results)

$$f_n \text{ converges uniformly to } f \text{ on } [-1, 2].$$

Therefore, by Theorem 5.10, and noting the derivative result

$$\frac{d}{dx} \left(\frac{e^x}{1 + e^x} \right) = \frac{e^x}{(1 + e^x)^2},$$

we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{-1}^2 f_n(x) dx &= \int_{-1}^2 \lim_{n \rightarrow \infty} f_n(x) dx = \int_{-1}^2 f(x) dx \\ &= \int_{-1}^2 \frac{e^x}{(1 + e^x)^2} dx \\ &= \left[\frac{e^x}{1 + e^x} \right]_{-1}^2 \\ &= \frac{e^2}{1 + e^2} - \frac{1}{1 + e} \\ &= \frac{e^3 - 1}{(1 + e)(1 + e^2)}. \end{aligned}$$

In summary,

$$\lim_{n \rightarrow \infty} \int_{-1}^2 \frac{e^x}{\left(1 + e^x + \frac{1}{4\sqrt{n}}\right)^2} dx = \frac{e^3 - 1}{(1 + e)(1 + e^2)}.$$

K.11. Relevant page(s) in the notes: 17, 69, 70, 79

(a) For each $n \in \mathbb{N}$ and $s > 1$ define

$$I(n, s) = \int_0^\infty x^{s-1} e^{-nx} dx.$$

Let

$$u = nx \quad \implies \quad x = \frac{u}{n} \quad \implies \quad \frac{dx}{du} = \frac{1}{n} \quad \implies \quad dx = \frac{du}{n}.$$

Also

$$x = 0 \quad \implies \quad u = 0 \quad \text{and} \quad x = \infty \quad \implies \quad u = \infty.$$

Replacement of x by u in the integral then leads to

$$I(n, s) = \int_0^\infty (u/n)^{s-1} e^{-u} du/n = \frac{1}{n^s} \int_0^\infty u^{s-1} e^{-u} du = \frac{\Gamma(s)}{n^s}.$$

Hence

$$\frac{1}{n^s} = \frac{I(n, s)}{\Gamma(s)}$$

from which the required result immediately follows.

(b) From the answer to part (a) and changing from $n \in \mathbb{N}$ to $k \in \mathbb{N}$ we have

$$\Gamma(s) \frac{1}{k^s} = I(k, s) \quad \text{for all } k \in \mathbb{N} \quad \text{and} \quad s > 1.$$

Therefore,

$$\begin{aligned} \Gamma(s) \sum_{k=1}^\infty \frac{1}{k^s} &= \sum_{k=1}^\infty \int_0^\infty x^{s-1} e^{-kx} dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n \int_0^\infty x^{s-1} e^{-kx} dx \\ &= \lim_{n \rightarrow \infty} \int_0^\infty \sum_{k=1}^n x^{s-1} e^{-kx} dx \\ &= \lim_{n \rightarrow \infty} \int_0^\infty f_n(x) dx \end{aligned}$$

where

$$f_n(x) = x^{s-1} \sum_{k=1}^n e^{-kx} \quad \text{for all } x \in \mathbb{R} \text{ and } n \in \mathbb{N}.$$

Since

$$f_{n+1}(x) - f_n(x) = x^{s-1} \sum_{k=1}^{n+1} e^{-kx} - x^{s-1} \sum_{k=1}^n e^{-kx} = x^{s-1} e^{-(n+1)x} \geq 0$$

we have

$$f_{n+1}(x) \geq f_n(x) \quad \text{for all } x \in \mathbb{R} \text{ and } n \in \mathbb{N}$$

and so by an extension of Theorem 5.9 to functions on $[0, \infty)$,

$$f_n \text{ converges uniformly to } f \text{ on } [0, \infty).$$

Therefore, by an extension of Theorem 5.10 to functions on $[0, \infty)$,

$$\lim_{n \rightarrow \infty} \int_0^\infty f_n(x) dx = \int_0^\infty \lim_{n \rightarrow \infty} f_n(x) dx = \int_0^\infty x^{s-1} \sum_{k=1}^\infty (e^{-x})^k dx.$$

Next note that

$$0 < x < \infty \implies 1 < e^x < \infty \implies 0 < e^{-x} < 1.$$

So, by the infinite geometric series result,

$$\sum_{k=1}^{\infty} (e^{-x})^k = \sum_{k=0}^{\infty} (e^{-x})^k - 1 = \frac{1}{1 - e^{-x}} - 1 = \frac{1}{e^x - 1}.$$

Hence,

$$\sum_{k=1}^{\infty} \frac{1}{k^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx. \quad (6)$$

Since the k on the left-hand side of (6) is a summation dummy variable, we can replace it by n to obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx.$$

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