UNIVERSITY OF TECHNOLOGY SYDNEY

School of Mathematical and Physical Sciences

35007 Real Analysis

EXERCISE SET I – Solutions

THEME: Series of Functions

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I.1. *Relevant page(s) in the notes:* 15, 16, 45, 46

If $a_n = (-1)^n/(n8^n)$ then some straightforward algebra leads to

$$\frac{a_{n+1}}{a_n} = \frac{-1}{8(1+1/n)}.$$

Therefore

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{8(1+1/n)}\right| = \frac{1}{8}.$$

Hence

$$|x|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \tfrac{1}{8}|x| < 1 \quad \text{for all} \quad |x| < 8$$

and so by Definition 3.28 the radius of convergence of the series

$$\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n8^n} \quad \text{is} \quad R = 8. \tag{1}$$

From Theorem 3.30 this series converges for all $|x| < 8$ and diverges for all $|x| > 8$.

Next we treat the boundary $x = -8$ and $x = 8$ cases. When $x = -8$ the series in (1) reduces to

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

which is the harmonic series and famously diverges. When $x = 8$ the series in (1) reduces to

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

which is the alternating harmonic series and converges. On assembly of all of these findings, the series in (1) converges if and only if

$$-8 < x \le 8.$$

[1] This document is less than a few months old which makes errors more likely. Updated versions with later dates may be released during the semester. Notifications will be sent via e-mail whenever an updated version is released.

I.2. $\boxed{\textit{Relevant page(s) in the notes: } 19, 68}$

The fact that $|\cos(y)| \le 1$ implies that $|\cos(y)|^5 = |\cos^5(y)| \le 1$ for all $y \in \mathbb{R}$. Hence

$$\left|\frac{\cos^5(n^{11}\,x)}{n\sqrt{n+9}}\right| \le M_n \quad \text{for all } x \in \mathbb{R}$$

where

$$M_n = \frac{1}{n\sqrt{n+9}}, \qquad n \in \mathbb{N}.$$

To prove that

$$\sum_{n=1}^{\infty} M_n \text{ converges} \quad \text{we use the fact that} \quad \sum_{n=1}^{\infty} n^{-3/2} \text{ converges}$$

and observe that

$$\lim_{n\to\infty} \frac{M_n}{n^{-3/2}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+9/n}} = 1.$$

By the Limit Comparison Test (Theorem 1.40), $\sum_{n=1}^{\infty} M_n$ converges. Application of the Weierstrass M-test (Theorem 5.7) then provides

$$\sum_{n=1}^{\infty} \frac{\cos^5(n^{11}\,x)}{n\sqrt{n+9}} \quad \text{is uniformly convergent on } \mathbb{R}.$$

I.3. $\boxed{\textit{Relevant page(s) in the notes: } 15, 16, 45, 46}$

(a) If $a_n = 1/n$ then some straightforward algebra leads to

$$\frac{a_{n+1}}{a_n} = \frac{1}{1+1/n}.$$

Therefore

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{1+1/n}\right| = 1.$$

Hence

$$|x| \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x| < 1 \quad \text{for all} \quad |x| < 1$$

and so by Definition 3.28 the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{is} \quad R = 1. \tag{2}$$

From Theorem 3.30 this series converges for all $|x| < 1$ and diverges for all $|x| > 1$. Next we treat the boundary $x = -1$ and $x = 1$ cases. When $x = -1$ the series in (2) reduces to

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

which is the alternating harmonic series which converges. When $x = 1$ the series in (2) reduces to

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

which is the harmonic series and famously diverges. On assembly of all of these findings, the series in (2) converges if and only if

$$-1 \le x < 1.$$

(b) Instead of (2) the series of interest now is

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} \tag{3}$$

Arguments very similar to those given in part (a) prove that the series in (3) converges if $|x| < 1$ and diverges if $|x| > 1$. For the $x = -1$ boundary case we have the harmonic series which diverges. For the $x = 1$ boundary case we have the alternating harmonic series which converges. Therefore, the series in (3) converges if and only if

$$-1 < x \le 1.$$

I.4. *Relevant page(s) in the notes:* 45, 46

If $a_n = (-1)^n/(2n)!$ then some straightforward algebra leads to

$$\frac{a_{n+1}}{a_n} = \frac{-1}{2(n+1)(2n+1)}.$$

Therefore

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left\{ \frac{1}{2(n+1)(2n+1)} \right\} = 0.$$

Hence

$$|x^2| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x^2| \times 0 = 0 < 1 \quad \text{for all} \quad x \in \mathbb{R}.$$

This is equivalent to saying

$$|x^2| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \quad \text{for all } |x| < \infty$$

and so by Definition 3.28 the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{is} \quad R = \infty.$$

From Theorem 3.30, this series converges for all $x \in \mathbb{R}$.

I.5. $\boxed{\textit{Relevant page(s) in the notes: } 16, 22, 45, 46}$

If $a_n = (n+13)^{5/2}/(3n+4)$ then some straightforward algebra leads to

$$\frac{a_{n+1}}{a_n} = \left(\frac{1+14/n}{1+13/n}\right)^{5/2}\left(\frac{3+4/n}{3+7/n}\right).$$

Therefore

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1+14/n}{1+13/n}\right|^{5/2}\lim_{n\to\infty}\left|\frac{3+4/n}{3+7/n}\right| = 1.$$

Hence, somewhat trivially,

$$|x|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x| < 1 \quad \text{for all} \quad |x| < 1$$

and so by Definition 3.28 the radius of convergence of the series

$$\sum_{n=0}^{\infty}\frac{(n+13)^{5/2}x^n}{3n+4} \quad \text{is} \quad R = 1. \tag{4}$$

From Theorem 3.30 this series converges for all $|x| < 1$ and diverges for all $|x| > 1$.

Next we treat the boundary $x = -1$ and $x = 1$ cases. When $x = -1$ the series in (4) reduces to

$$\sum_{n=0}^{\infty}\frac{(-1)^n(n+13)^{5/2}}{3n+4}. \tag{5}$$

Note that

$$\frac{(-1)^n(n+13)^{5/2}}{3n+4} = \begin{cases} -\dfrac{(n^{3/5}+13n^{-2/5})^{5/2}}{3+4n^{-1}}, & \text{for } n \text{ odd} \\[2ex] \dfrac{(n^{3/5}+13n^{-2/5})^{5/2}}{3+4n^{-1}}, & \text{for } n \text{ even.} \end{cases} \tag{6}$$

It is clear from this expression that the subsequence of the left-hand side of (6) corresponding to the odd values of n diverges to $-\infty$ and so

$$\liminf_{n\to\infty}\frac{(-1)^n(n+13)^{5/2}}{3n+4} = -\infty$$

whilst the subsequence corresponding to the even values of n diverges to ∞ and so

$$\limsup_{n\to\infty} = \frac{(-1)^n(n+13)^{5/2}}{3n+4} = \infty.$$

By Proposition 1.44 the sequence on the left-hand side of (6) does not converge to 0. Therefore, by Lemma 1.36, the sequence in (5) is divergent.

When $x = 1$ the series in (4) reduces to

$$\sum_{n=0}^{\infty}\frac{(n+13)^{5/2}}{3n+4} = \sum_{n=0}^{\infty}\frac{(n^{3/5}+13n^{-2/5})^{5/2}}{3+4n^{-1}} \tag{7}$$

which has also such that each term diverges. So by Lemma 1.36 the series in (7) is divergent.

Assembly of each of these findings leads to the series in the exercise converging if and only if

$$-1 < x < 1.$$

I.6. $\boxed{\textit{Relevant page(s) in the notes: } 22}$

The series of interest is

$$\sum_{n=0}^{\infty} a_n \quad \text{where} \quad a_n = \{\ln(n)\}^n x^n.$$

In the case of $x \neq 0$

$$|a_n|^{1/n} = \{|\ln(n)|^n |x|^n\}^{1/n} = |\ln(n)||x|.$$

Then, noting that $\ln(n) \geq 0$ for $n \in \mathbb{N}$,

$$\limsup_{n\to\infty} |a_n|^{1/n} = |x| \limsup_{n\to\infty} |\ln(n)| = |x| \limsup_{n\to\infty} \ln(n).$$

Since $\ln(n)$ is unbounded and monotone increasing no subsequence can converge to a finite limit. Therefore $\limsup_{n\to\infty} \ln(n) = \infty$. By the nth Root Test (Theorem 1.45)

$$\sum_{n=0}^{\infty} \{\ln(n)\}^n x^n \quad \text{is divergent for all } x \neq 0.$$

In case of $x = 0$ we have

$$\sum_{n=0}^{\infty} \{\ln(n)\}^n x^n = 0 + 0 + \ldots = 0$$

so the series converges if and only if $x = 0$.

I.7. $\boxed{\textit{Relevant page(s) in the notes: } 45, 46}$

If $a_n = n^2/n!$ then some straightforward algebra leads to

$$\frac{a_{n+1}}{a_n} = \left(1 + \frac{1}{n}\right)\left(\frac{1}{n+1}\right).$$

Therefore

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|1 + \frac{1}{n}\right| \lim_{n\to\infty} \left|\frac{1}{n+1}\right| = 0.$$

Hence

$$|x| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x| \times 0 = 0 < 1 \quad \text{for all} \quad x \in \mathbb{R}.$$

This is equivalent to saying

$$|x| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \quad \text{for all } |x| < \infty$$

and so by Definition 3.28 the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n^2 x^n}{n!} \quad \text{is} \quad R = \infty.$$

From Theorem 3.30, this series converges for all $x \in \mathbb{R}$.

I.8. $\boxed{\textit{Relevant page(s) in the notes: } 19, 68}$

Note that
$$e^x < e^x + 1 \qquad \Longrightarrow \qquad \frac{e^x}{1+e^x} < 1.$$
Also, since $e^x > 0$ we have $e^x/(1+e^x) > 0$. Hence
$$\left|\frac{e^x}{1+e^x}\right| < 1 \quad \text{for all } x \in \mathbb{R}$$
which implies that
$$\left|\frac{e^x(n+17)}{(1+e^x)(5n^4+12n^2+8)}\right| \le M_n \quad \text{for all } x \in \mathbb{R}$$
where
$$M_n \equiv \frac{n+17}{5n^4+12n^2+8}, \qquad n \in \mathbb{N}.$$
To prove that
$$\sum_{n=1}^{\infty} M_n \text{ converges} \quad \text{we use the fact that} \quad \sum_{n=1}^{\infty} \frac{1}{n^3} \text{ converges}$$
and observe that
$$\lim_{n\to\infty} \frac{M_n}{n^{-3}} = \lim_{n\to\infty} \frac{1+17/n}{5+12/n^2+8/n^4} = \frac{1}{5}.$$
By the Limit Comparison Test (Theorem 1.40), $\sum_{n=1}^{\infty} M_n$ converges. Application of the Weierstrass M-test (Theorem 5.7) then provides
$$\sum_{n=1}^{\infty} \frac{e^x(n+17)}{(1+e^x)(5n^4+12n^2+8)} \quad \text{is uniformly convergent on } \mathbb{R}.$$

I.9. $\boxed{\textit{Relevant page(s) in the notes: } 47}$

The series in the exercise can be written
$$\sum_{n=0}^{\infty} nx^n \quad \text{with } x = \tfrac{1}{2}. \tag{8}$$
To obtain a general expression for the power series in (8) we start with the geometric series result
$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad |x| < 1$$
and differentiate both sides:
$$\frac{d}{dx}\sum_{n=0}^{\infty} x^n = \frac{d}{dx}\{(1-x)^{-1}\} \quad \text{which leads to} \quad \sum_{n=0}^{\infty} nx^{n-1} = (1-x)^{-2}.$$
The last step is justified by Theorem 3.32 in the notes. Multiplication of both sides by x gives
$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}, \qquad |x| < 1.$$

Setting $x = \frac{1}{2}$ leads to

$$\sum_{n=0}^{\infty} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \frac{6}{64} + \ldots = \frac{1/2}{(1-1/2)^2} = 2.$$

I.10. *Relevant page(s) in the notes: 47*

The series in the exercise can be written

$$\sum_{n=0}^{\infty} n^2 x^n \quad \text{with } x = 1/17. \tag{9}$$

To obtain a general expression for the power series in (9) we start with the geometric series result

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad |x| < 1$$

and differentiate both sides

$$\frac{d}{dx}\sum_{n=0}^{\infty} x^n = \frac{d}{dx}\{(1-x)^{-1}\} \qquad \Longrightarrow \qquad \sum_{n=0}^{\infty} n x^{n-1} = \frac{d}{dx}\{(1-x)^{-1}\} \tag{10}$$

The last step is justified by Theorem 3.32 in the notes. Multiplication of both sides of (10) by x then gives

$$\sum_{n=0}^{\infty} n x^n = x\,\frac{d}{dx}\{(1-x)^{-1}\}, \quad |x| < 1.$$

Repetition of these same two steps then leads to

$$\sum_{n=0}^{\infty} n^2 x^n = x\,\frac{d}{dx}\left[x\frac{d}{dx}\{(1-x)^{-1}\}\right], \quad |x| < 1. \tag{11}$$

The function of x on the right-hand side of (11) is

$$\begin{aligned} x\,\frac{d}{dx}\left[x\frac{d}{dx}\{(1-x)^{-1}\}\right] &= x\,\frac{d}{dx}\left\{x(1-x)^{-2}\right\} \\ &= x\left\{(1-x)^{-2} + 2x(1-x)^{-3}\right\} \\ &= \frac{x(x+1)}{(1-x)^3}. \end{aligned}$$

Hence,

$$\sum_{n=0}^{\infty} n^2 x^n = \frac{x(x+1)}{(1-x)^3}, \quad |x| < 1. \tag{12}$$

Substitution of $x = 1/17$ into (12) then gives

$$\sum_{n=0}^{\infty} \frac{n^2}{17^n} = \frac{(1/17)(18/17)}{(16/17)^3} = \frac{153}{2048}.$$

I.11. $\boxed{\textit{Relevant page(s) in the notes: } 47}$

The series in the exercise can be written

$$\sum_{n=0}^{\infty} n^3 x^n \quad \text{with } x = 1/11. \tag{13}$$

To obtain a general expression for the power series in (13) we start with the geometric series result

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad |x| < 1,$$

and differentiate both sides

$$\frac{d}{dx}\sum_{n=0}^{\infty} x^n = \frac{d}{dx}\{(1-x)^{-1}\} \qquad \Longrightarrow \qquad \sum_{n=0}^{\infty} n x^{n-1} = \frac{d}{dx}\{(1-x)^{-1}\} \tag{14}$$

The last step is justified by Theorem 3.32 in the notes. Multiplication of both sides of (14) by x then gives

$$\sum_{n=0}^{\infty} n x^n = x\,\frac{d}{dx}\{(1-x)^{-1}\}, \qquad |x| < 1.$$

Repetition of these same two steps then leads to

$$\sum_{n=0}^{\infty} n^2 x^n = x\,\frac{d}{dx}\left[x\frac{d}{dx}\{(1-x)^{-1}\}\right], \qquad |x| < 1.$$

Another repetition of these same two steps then leads to

$$\sum_{n=0}^{\infty} n^3 x^n = x\,\frac{d}{dx}\left(x\,\frac{d}{dx}\left[x\frac{d}{dx}\{(1-x)^{-1}\}\right]\right), \qquad |x| < 1. \tag{15}$$

The function of x on the right-hand side of (15) is

$$\begin{aligned}
x\,\frac{d}{dx}\left(x\,\frac{d}{dx}\left[x\frac{d}{dx}\{(1-x)^{-1}\}\right]\right) &= x\,\frac{d}{dx}\left[x\,\frac{d}{dx}\left\{x(1-x)^{-2}\right\}\right] \\
&= x\,\frac{d}{dx}\left[x\left\{(1-x)^{-2} + 2x(1-x)^{-3}\right\}\right] \\
&= x\,\frac{d}{dx}\left\{\frac{x(x+1)}{(1-x)^3}\right\} \\
&= \frac{x\{(2x+1)(1-x)^3 + 3(1-x)^2 x(x+1)\}}{(1-x)^6} \\
&= \frac{x\{(2x+1)(1-x) + 3x(x+1)\}}{(1-x)^4} \\
&= \frac{x(x^2+4x+1)}{(1-x)^4}.
\end{aligned}$$

Hence,

$$\sum_{n=0}^{\infty} n^3 x^n = \frac{x(x^2+4x+1)}{(1-x)^4}, \qquad |x| < 1. \tag{16}$$

Substitution of $x = 1/11$ into (16) then gives

$$\sum_{n=0}^{\infty} \frac{n^3}{11^n} = \frac{(1/11)\{(1/121) + (4/11) + 1\}}{(10/11)^4} = \frac{913}{5000}.$$

I.12. *Relevant page(s) in the notes:* 45, 46, 47

The Taylor series of $f(x)$ about $a = \pi/3$ is, from Definition 3.33,

$$\begin{aligned} f(x) &= f(a + x - a) \\ &= f(a) + f^{(1)}(a)(x-a) + \frac{1}{2!}f^{(2)}(a)(x-a)^2 + \frac{1}{3!}f^{(3)}(a)(x-a)^3 + \ldots \end{aligned}$$

Since $f(x) = \sin(3x)\cos(3x)$, we have $f(a) = \sin(\pi)\cos(\pi) = 0$.

The first derivative of f is

$$f^{(1)}(x) = 3\cos(3x)\cos(3x) - 3\sin(3x)\sin(3x) = 3\{\cos^2(3x) - \sin^2(3x)\}$$

which leads to $f^{(1)}(a) = f^{(1)}(\pi/3) = 3\{\cos^2(\pi) - \sin^2(\pi)\} = 3$.

The second derivative of f is

$$f^{(2)}(x) = 3\big[2\cos(3x)\{-3\sin(3x)\} - 2\sin(3x)\{3\cos(3x)\}\big] = -18\sin(3x)\cos(3x) = -36f(x)$$

and so $f^{(2)}(a) = f^{(2)}(\pi/3) = 0$.

The third derivative is $f^{(3)}(x) = -36f^{(1)}(x)$ which leads to $f^{(3)}(a) = f^{(3)}(\pi/3) = -36 \times 3$.

The fourth derivative is $f^{(4)}(x) = -36f^{(2)}(x)$ which leads to $f^{(4)}(a) = f^{(4)}(\pi/3) = 0$.

The fifth derivative is $f^{(5)}(x) = -36f^{(3)}(x)$ which leads to $f^{(5)}(a) = f^{(5)}(\pi/3) = (-36)^2 \times 3$.

It is now apparent that

$$f^{(2n)}(\pi/3) = 0 \quad \text{and} \quad f^{(2n+1)}(\pi/3) = 3(-36)^n \quad \text{for each } n = 0, 1, 2, \ldots$$

Substituting these results into the above general Taylor series expression leads to

$$\sin(3x)\cos(3x) = 3\sum_{n=0}^{\infty} \frac{(-36)^n (x - \pi/3)^{2n+1}}{(2n+1)!}. \tag{17}$$

To determine the values of x for which the series in (17) converges, note that it can be written as

$$3(x - \pi/3)\sum_{n=0}^{\infty} a_n (x - \pi/3)^{2n} \quad \text{where} \quad a_n = (-36)^n/(2n+1)!.$$

Then some straightforward algebra leads to

$$\frac{a_{n+1}}{a_n} = \frac{-18}{(n+1)(2n+3)}.$$

Hence

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 18\lim_{n\to\infty}\left\{\frac{1}{(n+1)(2n+3)}\right\} = 0.$$

Therefore

$$|x-\pi/3|^2\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-\pi/3|^2\times 0 = 0 < 1 \quad \text{for all} \quad x\in\mathbb{R}.$$

This is equivalent to saying

$$|x-\pi/3|^2\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1 \quad \text{for all } |x-\pi/3| < \infty$$

and so by Definition 3.28 the radius of convergence of the series

$$\sum_{n=0}^{\infty} a_n(x-\pi/3)^{2n} \quad \text{is} \quad R=\infty.$$

From Theorem 3.30, the series in (17) converges for all $x\in\mathbb{R}$.

I.13. *Relevant page(s) in the notes:* 45, 46, 47

The Taylor series of $f(x)$ about 0 is, from Definition 3.33,

$$f(x) = f(0) + f^{(1)}(0)x + \frac{1}{2!}f^{(2)}(0)x^2 + \frac{1}{3!}f^{(3)}(0)x^3 + \ldots$$

Since $f(x)=\cos^2(x)$, we have $f(0)=\cos^2(0)=1$.

The first derivative of f is

$$f^{(1)}(x) = -2\cos(x)\sin(x) = -\sin(2x)$$

which leads to $f^{(1)}(0) = -\sin(0) = 0$.

The second derivative of f is

$$f^{(2)}(x) = -2\cos(2x)$$

and so $f^{(2)}(0) = -2$.

The third derivative of f is

$$f^{(3)}(x) = 4\sin(2x) = -2f^{(1)}(x)$$

and so $f^{(3)}(0) = 0$.

The fourth derivative of f is

$$f^{(4)}(x) = -4f^{(2)}(x) = 8\cos(2x)$$

and so $f^{(3)}(0) = 8$.

The fifth derivative of f is

$$f^{(5)}(x) = -2f^{(3)}(x) = -16\sin(2x)$$

and so $f^{(5)}(0) = 0$.

The sixth derivative of f is

$$f^{(6)}(x) = -2f^{(4)}(x) = -32\cos(2x)$$

and so $f^{(6)}(0) = -32$.

It is now apparent that

$$f^{(2n-1)}(0) = 0 \quad \text{and} \quad f^{(2n)}(0) = \tfrac{1}{2}(-4)^n \quad \text{for each } n = 0, 1, 2, \ldots$$

Substituting these results into the above general Taylor series expression leads to

$$\cos^2(x) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-4)^n x^{2n}}{(2n)!}. \tag{18}$$

To determine the values of x for which the series in (18) converges, note that it can be written as

$$\sum_{n=0}^{\infty} a_n x^{2n} \quad \text{where} \quad a_n = \tfrac{1}{2}(-4)^n/(2n)!.$$

Then some straightforward algebra leads to

$$\frac{a_{n+1}}{a_n} = \frac{-2}{(n+1)(2n+1)}.$$

Hence

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2\lim_{n\to\infty}\left\{\frac{1}{(n+1)(2n+1)}\right\} = 0.$$

Therefore,

$$|x|^2 \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|^2 \times 0 = 0 < 1 \quad \text{for all} \quad x \in \mathbb{R}.$$

This is equivalent to saying

$$|x|^2 \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1 \quad \text{for all } |x| < \infty$$

and so by Definition 3.28 the radius of convergence of the series

$$\sum_{n=0}^{\infty} a_n x^{2n} \quad \text{is} \quad R = \infty.$$

From Theorem 3.30, the series in (18) converges for all $x \in \mathbb{R}$.

I.14. *Relevant page(s) in the notes:* 45, 46, 47

The Taylor series of $f(x)$ about 0 is, from Definition 3.33,

$$f(x) = f(0) + f^{(1)}(0)x + \frac{1}{2!}f^{(2)}(0)x^2 + \frac{1}{3!}f^{(3)}(0)x^3 + \ldots$$

Since $f(x) = \sinh(x)$, we have $f(0) = \sinh(0) = \frac{1}{2}(e^0 - e^{-0}) = 0$.

The first derivative of f is

$$f^{(1)}(x) = \frac{d}{dx}\left\{\tfrac{1}{2}(e^x - e^{-x})\right\} = \tfrac{1}{2}(e^x + e^{-x}) = \cosh(x)$$

and so $f^{(1)}(0) = \cosh(0) = \tfrac{1}{2}(e^0 + e^{-0}) = 1$.

The second derivative of f is

$$f^{(2)}(x) = \frac{d}{dx}\left\{\tfrac{1}{2}(e^x + e^{-x})\right\} = \tfrac{1}{2}(e^x - e^{-x}) = \sinh(x)$$

and so $f^{(2)}(0) = \sinh(0) = 0$.

The third derivative of f is

$$f^{(3)}(x) = \frac{d}{dx}\sinh(x) = \cosh(x)$$

and so $f^{(3)}(0) = 1$.

The third derivative of f is

$$f^{(4)}(x) = \frac{d}{dx}\cosh(x) = \sinh(x)$$

and so $f^{(4)}(0) = 0$.

From this pattern, it is apparent that the Taylor series of $\sinh(x)$ about 0 is,

$$\sinh(x) = \sum_{n=1}^{\infty} \frac{x^{2n+1}}{(2n+1)!}. \tag{19}$$

To determine the values of x for which the series converges, note that it can be written as

$$x\sum_{n=0}^{\infty} a_n x^{2n} \quad \text{where} \quad a_n = \frac{1}{(2n+1)!}.$$

Then some straightforward algebra leads to

$$\frac{a_{n+1}}{a_n} = \frac{1}{2(n+1)(2n+3)}.$$

Hence

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \tfrac{1}{2}\lim_{n\to\infty}\left\{\frac{1}{(n+1)(2n+3)}\right\} = 0.$$

Therefore

$$|x|^2 \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|^2 \times 0 = 0 < 1 \quad \text{for all} \quad x \in \mathbb{R}.$$

This is equivalent to saying

$$|x|^2 \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1 \quad \text{for all } |x| < \infty$$

and so by Definition 3.28 the radius of convergence of the series

$$x\sum_{n=0}^{\infty} a_n x^{2n} \quad \text{is} \quad R = \infty.$$

From Theorem 3.30, the series in (19) converges for all $x \in \mathbb{R}$.

I.15. $\boxed{\textit{Relevant page(s) in the notes: } 15, 16, 45, 46, 47}$

The Taylor series of $f(x)$ about $a = 29$ is, from Definition 3.33,

$$\begin{aligned} f(x) &= f(29 + x - 29) \\ &= f(29) + f^{(1)}(29)(x-29) + \frac{1}{2!}f^{(2)}(29)(x-29)^2 + \frac{1}{3!}f^{(3)}(29)(x-29)^3 + \dots \end{aligned}$$

Since $f(x) = \ln(x)$, we have $f(29) = \ln(29)$.

The first derivative of f is

$$f^{(1)}(x) = x^{-1} \qquad \Longrightarrow \qquad f^{(1)}(29) = 1/29.$$

The second derivative of f is

$$f^{(2)}(x) = -x^{-2} \qquad \Longrightarrow \qquad f^{(2)}(29) = -1/29^2.$$

The third derivative is of f is

$$f^{(3)}(x) = 2x^{-3} \qquad \Longrightarrow \qquad f^{(3)}(29) = 2/29^3.$$

The fourth derivative is of f is

$$f^{(4)}(x) = -6x^{-4} \qquad \Longrightarrow \qquad f^{(4)}(29) = -6/29^4.$$

The fifth derivative is of f is

$$f^{(5)}(x) = 24x^{-5} \qquad \Longrightarrow \qquad f^{(5)}(29) = 24/29^5.$$

It is apparent that

$$f^{(n)}(29) = \frac{(-1)^{n-1}(n-1)!}{29^n} \quad \text{for each} \quad n \in \mathbb{N}.$$

Therefore, the Taylor series of $\ln(x)$ around $a = 29$ is

$$\ln(x) = \ln(29) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!(x-29)^n}{n!\,29^n} = \ln(29) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-29)^n}{n\,29^n}. \quad (20)$$

To determine the values of x for which the series in (20) converges, note that $\ln(x) - \ln(29)$ can be written as

$$\sum_{n=1}^{\infty} a_n(x-29)^n \quad \text{where} \quad a_n = \frac{(-1)^{n-1}}{n\,29^n}.$$

Then some straightforward algebra leads to

$$\frac{a_{n+1}}{a_n} = -\frac{n}{29(n+1)}.$$

Hence

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{29}\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right) = \frac{1}{29}.$$

Therefore,

$$|x-29| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x-29|(1/29) < 1 \quad \text{for all} \quad |x-29| < 29.$$

and so by Definition 3.28 the radius of convergence of the series is $R = 29$. From Theorem 3.30, the series converges for all $|x-29| < 29$ and diverges for all $|x-29| > 29$. Next we treat the boundary cases corresponding to $|x-29| = 29$, which simplify to $x = 0$ and $x = 58$. When $x = 0$ the series in (20) reduces to

$$\ln(29) - \sum_{n=1}^{\infty} \frac{1}{n}.$$

The second term is the harmonic series, which famously diverges. When $x = 58$ the series in (20) reduces to

$$\ln(29) - \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

The second term is the alternating harmonic series, which converges.

In summary, based on the above findings, the Taylor series

$$\ln(x) = \ln(29) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-29)^n}{n\,29^n} \quad \text{converges if and only if} \quad 0 < x \leq 58.$$

I.16. *Relevant page(s) in the notes:* 18, 19, 68

We have

$$f_n(x) = \frac{x}{n(1+nx^2)}.$$

(a) First note the result (equation (1.2)) that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{converges to} \quad \frac{\pi^2}{6}.$$

Next note that for all $x \neq 0$

$$\lim_{n\to\infty} \frac{f_n(x)}{b_n} = \lim_{n\to\infty} \left(\frac{xn^2}{n+n^2x^2}\right) = \lim_{n\to\infty} \left(\frac{x}{1/n+x^2}\right) = \frac{1}{x} \neq 0$$

so by the Limit Comparison Test (Theorem 1.40)

$$\sum_{n=1}^{\infty} f_n(x) \quad \text{converges for all} \quad x \neq 0. \tag{21}$$

Since $f_n(0) = 0$ for all $n \in \mathbb{N}$ then, trivially,

$$\sum_{n=1}^{\infty} f_n(x) \quad \text{converges for} \quad x = 0. \tag{22}$$

Combining (21) and (22) we have

$$\sum_{n=1}^{\infty} f_n(x) \quad \text{converges pointwise for all } x \in \mathbb{R}.$$

(b) $\boxed{\textit{This is publication-quality proof. See † at the end of this document for a full explanation.}}$

Define the sequence $M_n = 1/(2n^{3/2})$, $n \in \mathbb{N}$, and note that for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$

$$\begin{aligned} M_n - |f_n(x)| &= \frac{1}{2n^{3/2}} - \frac{|x|}{n + n^2x^2} \\ &= \frac{n + n^2x^2 - 2n^{3/2}|x|}{2n^{3/2}(n + n^2x^2)} \\ &= \frac{nx^2 - 2\sqrt{n}|x| + 1}{\sqrt{n}(n + n^2x^2)} \\ &= \frac{\left(\sqrt{n}|x| - 1\right)^2}{\sqrt{n}(n + n^2x^2)} \\ &\geq 0. \end{aligned}$$

Therefore

$$|f_n(x)| \leq M_n \quad \text{for all } x \in \mathbb{R} \text{ and } n \in \mathbb{N}.$$

Since

$$\sum_{n=1}^{\infty} M_n = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \quad \text{is convergent,}$$

by the Weierstrass M-test (Theorem 5.7)

$$\sum_{n=1}^{\infty} f_n(x) \quad \text{converges uniformly over } \mathbb{R}.$$

Preliminary background working that led to obtaining an effective M_n

Note that

$$f_n'(x) = \frac{d}{dx}\frac{x}{n(1+nx^2)} = \frac{n(1+nx^2) - 2n^2x^2}{n^2(1+nx^2)^2} = \frac{1-nx^2}{n(1+nx^2)^2}.$$

Therefore

$$f_n'(x) = 0 \quad \Longleftrightarrow \quad 1 - nx^2 = 0 \quad \Longleftrightarrow \quad x = 1/\sqrt{n}$$

so f_n has a stationary point at $x = 1/\sqrt{n}$. Since

$$f_n''(x) = \frac{d}{dx}\frac{1-nx^2}{n(1+nx^2)^2} = \frac{2x(nx^2-3)}{(nx^2+1)^3}$$

we have

$$f_n''(1/\sqrt{n}) = \frac{2(1-3)}{(1+1)^3\sqrt{n}} = \frac{-1}{2\sqrt{n}} < 0$$

which means that f_n has at least a local maximum at $x = 1/\sqrt{n}$. Therefore

$$f_n(1/\sqrt{n}) = \frac{1}{2n^{3/2}}$$

is likely to be a good choice for M_n.

† According to the lecturer's experiences in publishing proofs in mathematics and statistics journals over many years, preliminary background working **should not be included** in the proof. This is the convention in esteemed journals such as *The Annals of Mathematics* and *The Annals of Statistics*. Many web-sites and some books **do not follow** publication-quality conventions and tend to include such preliminary background working. Subject participants are encouraged to learn about and use publication-quality versions of mathematical proofs.

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