UNIVERSITY OF TECHNOLOGY SYDNEY

School of Mathematical and Physical Sciences

35007 Real Analysis

EXERCISE SET H – Solutions

THEME: L'Hôpital's Rule

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H.1. *Relevant page(s) in the notes:* 39

Since $\tan^{-1}(0) = 0$ both the numerator and denominator limits are 0, so by L'Hôpital's rule the required limit equals

$$\lim_{x\to 0} \frac{\frac{d}{dx}\tan^{-1}(109x)}{\frac{d}{dx}\left(42x + 71x\sqrt{x}\right)} = \lim_{x\to 0} \frac{109/(11881x^2+1)}{42 + 213\sqrt{x}/2} = \frac{109}{42}.$$

H.2. *Relevant page(s) in the notes:* 39, 40

Let

$$f(x) = \ln(x) \quad \text{and} \quad g(x) = \ln(x^3 + x).$$

Since

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$$

L'Hôpital's rule applies and we have

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{1/x}{(3x^2+1)/(x^3+x)} = \lim_{x\to\infty} \frac{x^2+1}{3x^2+1} = \lim_{x\to\infty} \frac{1+1/x^2}{3+1/x^2} = \frac{1}{3}.$$

H.3. *Relevant page(s) in the notes:* 39, 40

Since $\sin(0) = \tan(0) = 0$ both the numerator and denominator limits are 0, so by L'Hôpital's rule

$$\lim_{x\to 0} \frac{\sin^2(x)}{\tan(x^2)} = \lim_{x\to 0} \frac{\frac{d}{dx}\sin^2(x)}{\frac{d}{dx}\tan(x^2)} = \lim_{x\to 0} \frac{2\sin(x)\cos(x)}{\sec^2(x^2)(2x)} = \lim_{x\to 0} \frac{\sin(x)\cos(x)\cos^2(x^2)}{x}.$$

Once again, the numerator and denominator limits are 0 so by another application of L'Hôpital's rule

$$\lim_{x\to 0} \frac{\sin(x)\cos(x)\cos^2(x^2)}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}\big\{\sin(x)\cos(x)\cos^2(x^2)\big\}}{\frac{d}{dx}x} = \lim_{x\to 0} \frac{d}{dx}\big\{\sin(x)\cos(x)\cos^2(x^2)\big\}.$$

[1] This document is less than a few months old which makes errors more likely. Updated versions with later dates may be released during the semester. Notifications will be sent via e-mail whenever an updated version is released.

Since

$$\frac{d}{dx}\Big\{\sin(x)\cos(x)\cos^2(x^2)\Big\} = \cos^2(x)\cos^2(x^2) - \sin^2(x)\cos^2(x^2)$$
$$-2\cos(x^2)\sin(x^2)(2x)\sin(x)\cos(x).$$

and, noting that $\cos(0) = 1$, the required limit is

$$\lim_{x\to 0}\Big\{\cos^2(x)\cos^2(x^2) - \sin^2(x)\cos^2(x^2) - 2\cos(x^2)\sin(x^2)(2x)\sin(x)\cos(x)\Big\} = 1.$$

H.4. $\boxed{\textit{Relevant page(s) in the notes: } 39}$

Since $2^{14} = 16348$ both the numerator and denominator limits are 0, so by L'Hôpital's rule

$$\lim_{x\to 16348}\frac{x^{1/14}-2}{x-16348} = \lim_{x\to 16348}\frac{\frac{d}{dx}\big(x^{1/14}-2\big)}{\frac{d}{dx}\big(x-16348\big)} = \lim_{x\to 16348}\frac{x^{-13/14}}{14} = \frac{16348^{-13/14}}{14}.$$

Noting that $16348^{1/14} = 2$ we then obtain the limit in the exercise equalling

$$\frac{1}{14\times 2^{13}} = \frac{1}{114688}.$$

H.5. $\boxed{\textit{Relevant page(s) in the notes: } 39, 40}$

Note that for all $x > 0$ we have

$$x\ln(x) = -\frac{-\ln(x)}{(1/x)} = \frac{f(x)}{g(x)} \quad \text{where} \quad f(x) \equiv -\ln(x) \text{ and } g(x) \equiv 1/x.$$

Then

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+}\{-\ln(x)\} = \infty \quad \text{and} \quad \lim_{x\to 0+} g(x) = \lim_{x\to 0+}(1/x) = \infty.$$

By L'Hôpital's rule

$$\lim_{x\to 0^+}\frac{f(x)}{g(x)} = \lim_{x\to 0^+}\frac{f'(x)}{g'(x)} = \lim_{x\to 0^+}\frac{(-x^{-1})}{(-x^{-2})} = \lim_{x\to 0^+} x = 0.$$

In summary,

$$\lim_{x\to 0^+} x\ln(x) = 0.$$

H.6. $\boxed{\textit{Relevant page(s) in the notes: } 39}$

Since $\sin(0) = 0$ both the numerator and denominator limits are 0, so by L'Hôpital's rule

$$\lim_{x\to 0}\frac{e^x\sin(x) - x(1+x)}{x^3} = \lim_{x\to 0}\frac{\frac{d}{dx}\{e^x\sin(x) - x(1+x)\}}{\frac{d}{dx}x^3} = \lim_{x\to 0}\frac{e^x\{\sin(x)+\cos(x)\} - 2x - 1}{3x^2}.$$

Once again, noting that $\cos(0) = 1$, the numerator and denominator limits are 0 so by another application of L'Hôpital's rule

$$\lim_{x\to 0} \frac{e^x\{\sin(x) + \cos(x)\} - 2x - 1}{3x^2} = \lim_{x\to 0} \frac{\frac{d}{dx}\left[e^x\{\sin(x) + \cos(x)\} - 2x - 1\right]}{\frac{d}{dx}(3x^2)}$$
$$= \lim_{x\to 0} \frac{2e^x\cos(x) - 2}{6x}.$$

For the third time, the numerator and denominator limits are 0 so by yet another application of L'Hôpital's rule

$$\lim_{x\to 0} \frac{2e^x\cos(x) - 2}{6x} \lim_{x\to 0} \frac{\frac{d}{dx}\{2e^x\cos(x) - 2\}}{\frac{d}{dx}(6x)} = \lim_{x\to 0} \frac{2e^x\{\cos(x) - \sin(x)\}}{6} = \frac{1}{3}.$$

In summary,

$$\lim_{x\to 0} \frac{e^x\sin(x) - x(1+x)}{x^3} = \frac{1}{3}.$$

H.7. *Relevant page(s) in the notes:* 39

Since $\sin(0) = 0$ both the numerator and denominator limits are 0, so by L'Hôpital's rule

$$\lim_{x\to 0} \frac{\sin^{947}(x)}{\sin(x^{947})} = \lim_{x\to 0} \frac{\frac{d}{dx}\sin^{947}(x)}{\frac{d}{dx}\sin(x^{947})}$$
$$= \lim_{x\to 0} \frac{947\sin^{946}(x)\cos(x)}{\cos(x^{947})947x^{946}}$$
$$= \lim_{x\to 0} \left\{\left(\frac{\sin(x)}{x}\right)^{946} \frac{\cos(x)}{\cos(x^{947})}\right\}$$
$$= \left\{\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right)\right\}^{946} \lim_{x\to 0} \frac{\cos(x)}{\cos(x^{947})}$$
$$= \left\{\lim_{x\to 0}\left(\frac{\frac{d}{dx}\sin(x)}{\frac{d}{dx}(x)}\right)\right\}^{946} \frac{\cos(0)}{\cos(0)}$$
$$= \left\{\lim_{x\to 0}\left(\frac{\cos(x)}{1}\right)\right\}^{946}$$
$$= \cos^{946}(0)$$
$$= 1.$$

H.8. *Relevant page(s) in the notes:* 39

First note that

$$\lim_{x\to 11+} \frac{\ln(x-11)}{\ln(x^2-121)} = \lim_{x\to 11+} \frac{-\ln(x-11)}{-\ln(x^2-121)}.$$

Since

$$\lim_{x\to 11+}\{-\ln(x-11)\}=\infty \quad \text{and} \quad \lim_{x\to 11+}\{-\ln(x^2-121)\}=\infty$$

L'Hôpital's rule applies. Then

$$\begin{aligned}
\lim_{x\to 11+}\frac{\ln(x-11)}{\ln(x^2-121)} &= \lim_{x\to 11+}\frac{\frac{d}{dx}\ln(x-11)}{\frac{d}{dx}\ln(x^2-121)} \\
&= \lim_{x\to 11+}\frac{1/(x-11)}{2x/(x^2-121)} \\
&= \lim_{x\to 11+}\frac{(x-11)(x+11)}{2x(x-11)} \\
&= \lim_{x\to 11+}\frac{x+11}{2x} \\
&= 1.
\end{aligned}$$

H.9. $\boxed{\textit{Relevant page(s) in the notes: } 39}$

Since

$$\lim_{x\to 0}(39^x-22^x)=0 \quad \text{and} \quad \lim_{x\to 0} x=0$$

L'Hôpital's rule applies. Then

$$\begin{aligned}
\lim_{x\to 0}\frac{39^x-22^x}{x} &= \lim_{x\to 0}\frac{\exp\{x\ln(39)\}-\exp\{x\ln(22)\}}{x} \\
&= \lim_{x\to 0}\frac{\frac{d}{dx}[\exp\{x\ln(39)\}-\exp\{x\ln(22)\}]}{\frac{d}{dx}x} \\
&= \lim_{x\to 0}\frac{\ln(39)\exp\{x\ln(39)\}-\ln(22)\exp\{x\ln(22)\}}{1} \\
&= \ln\left(\frac{39}{22}\right).
\end{aligned}$$

H.10. $\boxed{\textit{Relevant page(s) in the notes: } 39, 40}$

Since both $\ln\big(\ln(x)\big)$ and $\sqrt{x}$ approach ∞ as $x\to\infty$, L'Hôpital's rule applies and we have

$$\begin{aligned}
\lim_{x\to\infty}\frac{\ln\big(\ln(x)\big)}{\sqrt{x}} &= \lim_{x\to\infty}\frac{\frac{d}{dx}\ln\big(\ln(x)\big)}{\frac{d}{dx}(x^{1/2})} \\
&= \lim_{x\to\infty}\frac{\{1/\ln(x)\}(1/x)}{\frac{1}{2}x^{-1/2}} \\
&= \lim_{x\to\infty}\left\{\frac{2}{\sqrt{x}\,\ln(x)}\right\} \\
&= 0.
\end{aligned}$$

H.11. $\boxed{\textit{Relevant page(s) in the notes: } 39}$

First note that

$$\lim_{x\to 1+}\{(x-1)\tan(\pi\,x/2)\} = \lim_{x\to 1+}\frac{(x-1)\sin(\pi\,x/2)}{\cos(\pi\,x/2)}.$$

Since $\cos(\pi/2)=0$ both the numerator and denominator limits are 0, so by L'Hôpital's rule the required limit is

$$\begin{aligned}
\lim_{x\to 1+}\frac{\frac{d}{dx}\{(x-1)\sin(\pi\,x/2)\}}{\frac{d}{dx}\cos(\pi\,x/2)} &= \lim_{x\to 1+}\frac{\sin(\pi\,x/2)+(x-1)\cos(\pi\,x/2)(\pi/2)}{-\sin(\pi\,x/2)(\pi/2)}\\
&= -\lim_{x\to 1+}\frac{\sin(\pi\,x/2)}{\sin(\pi\,x/2)(\pi/2)} - \lim_{x\to 1+}\frac{(x-1)\cos(\pi\,x/2)}{\sin(\pi\,x/2)}\\
&= -\lim_{x\to 1+}\frac{1}{(\pi/2)} - \lim_{x\to 1+}\frac{(x-1)\cos(\pi\,x/2)}{\sin(\pi\,x/2)}\\
&= -\frac{2}{\pi}+0\\
&= -\frac{2}{\pi}.
\end{aligned}$$

H.12. $\boxed{\textit{Relevant page(s) in the notes: } 39, 40}$

First note that

$$\begin{aligned}
\frac{x^4}{x^2-1}-\frac{x^4}{x^2+1} &= \frac{x^4(x^2+1)}{(x^2-1)(x^2+1)}-\frac{x^4(x^2-1)}{(x^2+1)(x^2-1)}\\
&= \frac{x^6+x^4-x^6+x^4}{x^4-1}\\
&= \frac{2x^4}{x^4-1}.
\end{aligned}$$

Therefore, the required limit is

$$\lim_{x\to\infty}\frac{2x^4}{x^4-1}.$$

Both the numerator and denominator limits are ∞, so by L'Hôpital's rule the required limit equals

$$\lim_{x\to\infty}\frac{\frac{d}{dx}(2x^4)}{\frac{d}{dx}(x^4-1)} = \lim_{x\to\infty}\frac{8x^3}{4x^3} = \lim_{x\to\infty}\frac{8}{4} = 2.$$

H.13. $\boxed{\textit{Relevant page(s) in the notes: } 25, 39, 40}$

Consider the function $f:(0,\infty)\to\mathbb{R}$ given by

$$f(x)=x^x,\quad x>0.$$

Then

$$f(x)=\exp\big(x\ln(x)\big)\quad\Longrightarrow\quad \lim_{x\to 0+}f(x)=\exp\Big(\lim_{x\to 0+}x\ln(x)\Big)\tag{1}$$

Note that for all $x > 0$ we have

$$x\ln(x) = -\frac{-\ln(x)}{(1/x)} = \frac{g(x)}{h(x)} \quad \text{where} \quad g(x) \equiv -\ln(x) \text{ and } h(x) \equiv 1/x.$$

Then

$$\lim_{x\to 0+} g(x) = \lim_{x\to 0+}\{-\ln(x)\} = \infty \quad \text{and} \quad \lim_{x\to 0+} h(x) = \lim_{x\to 0+}(1/x) = \infty$$

By L'Hôpital's rule

$$\lim_{x\to 0+} x\ln(x) = \lim_{x\to 0+}\frac{g(x)}{h(x)} = \lim_{x\to 0^+}\frac{g'(x)}{h'(x)} = \lim_{x\to 0^+}\frac{(-x^{-1})}{(-x^{-2})} = \lim_{x\to 0^+} x = 0.$$

Substitution into (**??**) then gives

$$\lim_{x\to 0+} f(x) = \exp(0) = 1.$$

Hence

$$\lim_{x\to 0+} x^x = 1$$

and, in terms of limit theory, the correct interpretation of 0^0 is

$$0^0 = 1.$$

H.14. $\boxed{\textit{Relevant page(s) in the notes: } 39}$

The required limit is

$$\lim_{x\to 0}\frac{f(x)}{g(x)}$$

where

$$f(x) = 1 + \tfrac{1}{2}x^2 - \sqrt{1+x^2} \quad \text{and} \quad g(x) = \big(\cos(x) - e^{x^2}\big)\sin(x^2).$$

Since $\sin(0) = 0$ both the numerator and denominator limits are 0, so by L'Hôpital's rule the required limit equals

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)}. \tag{2}$$

The numerator in (**??**) is

$$f'(x) = \frac{d}{dx}\big\{1 + \tfrac{1}{2}x^2 - (1+x^2)^{1/2}\big\} = x - x(1+x^2)^{-1/2}$$

and the denominator in (**??**) is

$$g'(x) = \frac{d}{dx}\big\{\big(\cos(x) - e^{x^2}\big)\sin(x^2)\big\} = 2x\big(\cos(x) - e^{x^2}\big)\cos(x^2) - \big(\sin(x) + 2xe^{x^2}\big)\sin(x^2).$$

The numerator and denominator limits in (**??**) are 0, so another application of L'Hôpital's rule leads to the required limit equalling

$$\lim_{x\to 0}\frac{f''(x)}{g''(x)}. \tag{3}$$

The numerator in (**??**) is

$$f''(x) = 1 - (1 + x^2)^{-3/2}.$$

The denominator in (**??**) is

$$g''(x) = -2\cos(x^2)\big\{e^{x^2}(4x^2+1) + 2x\sin(x) - \cos(x)\big\} - \sin(x^2)\big\{2e^{x^2} + \cos(x)(4x^2+1)\big\}.$$

The numerator and denominator limits in (**??**) are 0, so yet another application of L'Hôpital's rule leads to the required limit equalling

$$\lim_{x\to 0} \frac{f'''(x)}{g'''(x)}. \tag{4}$$

The numerator in (**??**) is

$$f'''(x) = 3x(1 + x^2)^{-5/2}.$$

The denominator in (**??**) is

$$\begin{aligned} g'''(x) = {} & \sin(x^2)\big\{12x^2\sin(x) + 16x^3e^{x^2} + \sin(x) - 12x\cos(x)\big\} \\ & -2\cos(x^2)\big\{4xe^{x^2}(2x^2+3) + x(4x^2+3)\cos(x) + 3\sin(x)\big\}. \end{aligned}$$

Remembering that $\sin(0) = 0$, the numerator and denominator limits in (**??**) are 0, so a fourth application of L'Hôpital's rule leads to the required limit equalling

$$\lim_{x\to 0} \frac{f''''(x)}{g''''(x)}. \tag{5}$$

The numerator in (**??**) is

$$f''''(x) = \frac{3(1-4x^2)}{(1+x^2)^{7/2}}.$$

The denominator in (**??**) is

$$\begin{aligned} g''''(x) = {} & 4(4x^2+1)\cos(x^2)\{2x\sin(x) - 6e^{x^2} - 3\cos(x)\} \\ & + \sin(x^2)\Big[16x\big\{2xe^{x^2}(2x^2+3) + 3\sin(x)\big\} + (16x^4 + 24x^2 - 11)\cos(x)\Big]. \end{aligned}$$

Now observe that

$$\lim_{x\to 0} f''''(x) = \frac{3(1-0)}{(1+0)^{7/2}} = 3 \quad \text{and} \quad \lim_{x\to 0} g''''(x) = 4(1)(0-6-3) + 0 = -36.$$

and so

$$\lim_{x\to 0} \frac{f''''(x)}{g''''(x)} = \frac{3}{-36} = -\frac{1}{12}.$$

Therefore, in summary,

$$\lim_{x\to 0} \left\{ \frac{1 + \frac{1}{2}x^2 - \sqrt{1+x^2}}{\big(\cos(x) - e^{x^2}\big)\sin(x^2)} \right\} = -\frac{1}{12}.$$

H.15. $\boxed{\textit{Relevant page(s) in the notes: } 39}$

First note that

$$\cot(x)^{\sin(x)} = \exp\left\{ \ln\left(\cot(x)^{\sin(x)} \right) \right\} = \exp\left\{ \sin(x)\ \ln\left(\cot(x) \right) \right\} = \exp\left\{ \frac{\ln\left(\dfrac{\cos(x)}{\sin(x)} \right)}{1/\sin(x)} \right\}.$$

Therefore,

$$\lim_{x\to 0+} \cot(x)^{\sin(x)} = \exp\left[\lim_{x\to 0+} \left\{ \frac{\ln\left(\dfrac{\cos(x)}{\sin(x)} \right)}{1/\sin(x)} \right\} \right]. \tag{6}$$

To evaluate the limit inside the $\exp$ function on the right-hand side of (**??**) we note that both the numerator and denominator approach ∞ and, so, L'Hôpital's rule applies. Thus

$$\lim_{x\to 0+} \left\{ \frac{\ln\left(\dfrac{\cos(x)}{\sin(x)} \right)}{1/\sin(x)} \right\} = \lim_{x\to 0+} \left[\frac{\frac{d}{dx} \ln\left(\dfrac{\cos(x)}{\sin(x)} \right)}{\frac{d}{dx}\{\sin(x)^{-1}\}} \right]. \tag{7}$$

The numerator derivative is, with use of the $\sin^2(a) + \cos^2(a) = 1$ identity,

$$\frac{d}{dx} \ln\left(\frac{\cos(x)}{\sin(x)} \right) = \left\{ \frac{\sin(x)}{\cos(x)} \right\} \left\{ \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} \right\} = -\frac{1}{\sin(x)\cos(x)}$$

The denominator derivative is

$$\frac{d}{dx}\{\sin(x)^{-1}\} = -\sin(x)^{-2}\cos(x).$$

Therefore, from L'Hôpital's rule (Theorem 3.13), the limit on the right-hand side of (**??**) becomes

$$\lim_{x\to 0+} \left[\frac{-1/\{\sin(x)\cos(x)\}}{-\sin(x)^{-2}\cos(x)} \right] = \lim_{x\to 0+} \left\{ \frac{\sin(x)}{\cos^2(x)} \right\} = 0.$$

Substitution into (**??**) then gives

$$\lim_{x\to 0+} \cot(x)^{\sin(x)} = e^0 = 1.$$

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