UNIVERSITY OF TECHNOLOGY SYDNEY

School of Mathematical and Physical Sciences

35007 Real Analysis

EXERCISE SET G – Solutions

THEME: Derivatives Theory

Date of this version: **22nd July 2026**[1]

G.1. *Relevant page(s) in the notes:* 30, 31

By the limit definition of a derivative (Definition 3.1)

$$\begin{aligned} f'(-3) &= \lim_{h\to 0} \frac{f(-3+h) - f(-3)}{h} \\ &= \lim_{h\to 0} \frac{1/(-3+h)^2 - 1/(-3)^2}{h} \\ &= \lim_{h\to 0} \frac{9-(h-3)^2}{9h(h-3)^2} \\ &= \lim_{h\to 0} \frac{(6-h)h}{9h(h-3)^2} \\ &= \lim_{h\to 0} \frac{6-h}{9(h-3)^2} \\ &= \frac{6}{9(-3)^2} \\ &= \frac{2}{27}. \end{aligned}$$

G.2. *Relevant page(s) in the notes:* 30, 31

Note that for any $x, h \in \mathbb{R}$

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^3 - x^3}{h} \\ &= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= 3x^2 + 3xh + h^2. \end{aligned}$$

Therefore, for all $x \in \mathbb{R}$,

$$f'(x) = \lim_{h\to 0}(3x^2 + 3xh + h^2) = 3x^2.$$

[1] This document is less than a few months old which makes errors more likely. Updated versions with later dates may be released during the semester. Notifications will be sent via e-mail whenever an updated version is released.

G.3. $\boxed{\textit{Relevant page(s) in the notes: } 30, 31}$

Note that for any $x, h \in \mathbb{R}$ such that $x > 0$ and $x + h > 0$

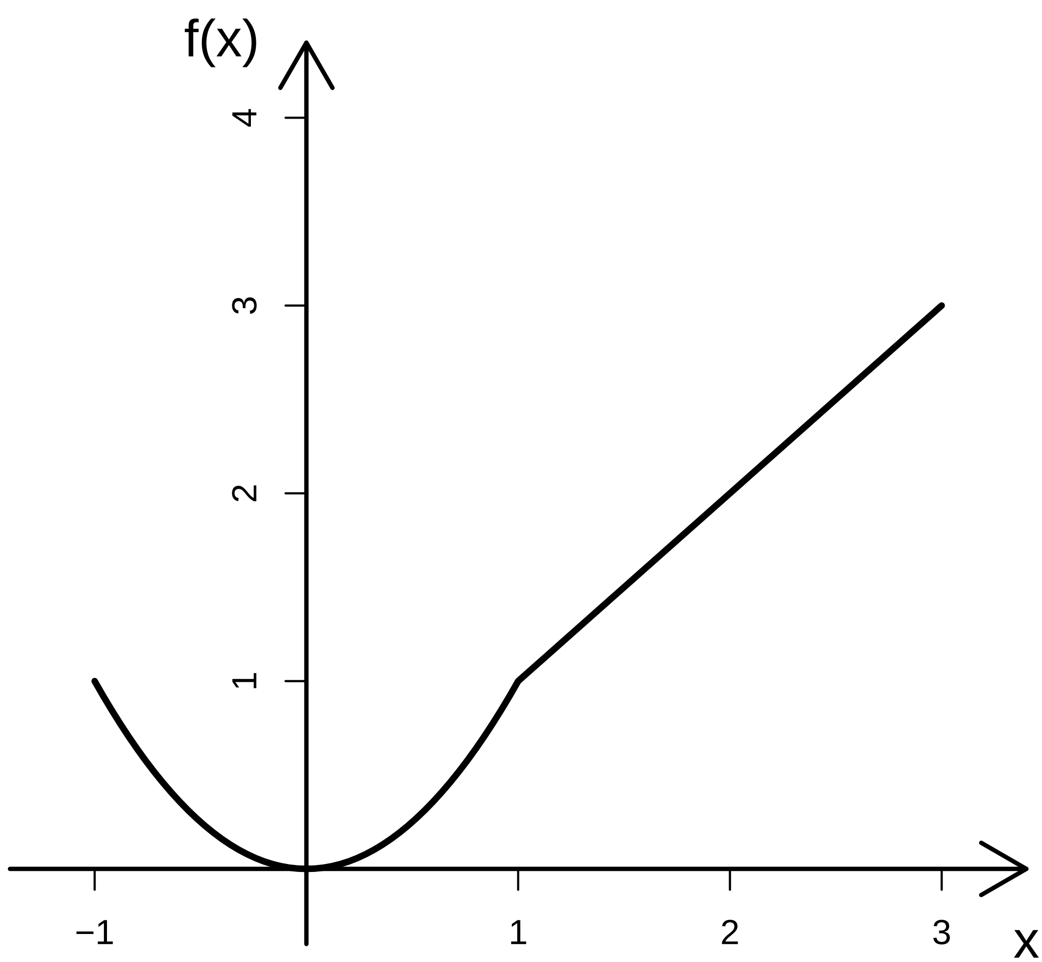
$$\begin{aligned}\frac{\sqrt{x+h}-\sqrt{x}}{h} &= \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}\\ &= \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}\\ &= \frac{1}{\sqrt{x+h}+\sqrt{x}}.\end{aligned}$$

Therefore, for all $x > 0$,

$$f'(x) = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

G.4. $\boxed{\textit{Relevant page(s) in the notes: } 25, 30, 31}$

(a) The graph of f over the interval $[-1, 3]$ is shown in the accompanying figure.



(b) We treat the following two cases separately:

(i) $x < 1$ and (ii) $x > 1$.

Case (i). Throughout Case (i) we have $x < 1$, which implies that $f(x) = x^2$. The derivative of f at x based on the left limit is

$$\lim_{h \to 0-} \frac{f(x+h) - x^2}{h}. \tag{1}$$

The limit in (**??**) involves h approaching 0 from the left, which is such that

$$h < 0 \quad \Longrightarrow \quad x + h < 1 \quad \Longrightarrow \quad f(x+h) = (x+h)^2.$$

Therefore, the limit in (**??**) is

$$\lim_{h \to 0-} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0-} (2x + h) = 2x.$$

The derivative of f at x based on the right limit is

$$\lim_{h \to 0+} \frac{f(x+h) - x^2}{h}. \tag{2}$$

The limit in (**??**) involves h approaching 0 from the right, which is such that $h > 0$ but becoming arbitrarily small. Remembering that $x < 1$, for small enough positive h we also have $x + h < 1$ which in implies $f(x+h) = (x+h)^2$. Therefore, the limit in (**??**) is

$$\lim_{h \to 0+} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0+} (2x + h) = 2x.$$

Since,

$$\lim_{h \to 0-} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0+} \frac{(x+h)^2 - f(x)}{h} = 2x$$

f is differentiable for all $x < 1$ (Definition 3.1).

Case (ii) Throughout Case (ii) we have $x > 1$, which implies that $f(x) = 2x$. The derivative of f at x based on the right limit is

$$\lim_{h \to 0+} \frac{f(x+h) - 2x}{h}. \tag{3}$$

The limit in (**??**) involves h approaching 0 from the right, which is such that

$$h > 0 \quad \Longrightarrow \quad x + h > 1 \quad \Longrightarrow \quad f(x+h) = 2(x+h).$$

Therefore, the limit in (**??**) is

$$\lim_{h \to 0+} \frac{2(x+h) - 2x}{h} = \lim_{h \to 0-} 2 = 2.$$

The derivative of f at x based on the left limit is

$$\lim_{h \to 0-} \frac{f(x+h) - 2x}{h}. \tag{4}$$

The limit in (**??**) involves h approaching 0 from the left, which is such that $h < 0$ but becoming arbitrarily small. Remembering that $x > 1$, for small enough negative h we also have $x+h>1$ which in implies $f(x+h)=2(x+h)$. Therefore, the limit in (**??**) is

$$\lim_{h\to 0-}\frac{2(x+h)-2x}{h}=\lim_{h\to 0-}2=2.$$

Since,

$$\lim_{h\to 0-}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0+}\frac{(x+h)^2-f(x)}{h}=2$$

f is differentiable for all $x>1$ (Definition 3.1).

Combining the conclusions from Case (i) and Case (ii) we have f is differentiable for all $x\neq 1$.

(c) The derivative of f at $x=1$ based on the left limit is

$$\lim_{h\to 0-}\frac{f(1+h)-f(1)}{h}. \tag{5}$$

The limit in (**??**) involves h approaching 0 from the left, which is such that

$$h<0 \quad\Longrightarrow\quad 1+h<1 \quad\Longrightarrow\quad f(1+h)=(1+h)^2.$$

Therefore, the limit in (**??**) is

$$\begin{aligned}\lim_{h\to 0-}\frac{(1+h)^2-1}{h}&=\lim_{h\to 0-}\frac{(1+h)^2-1}{h}\\&=\lim_{h\to 0-}\frac{1+2h+h^2-1}{h}\\&=\lim_{h\to 0-}(2+h)\\&=2.\end{aligned}$$

The derivative of f at $x=1$ based on the right limit is

$$\lim_{h\to 0+}\frac{f(1+h)-f(1)}{h}. \tag{6}$$

This limit in (**??**) involves h approaching 0 from the right, which is such that

$$h>0 \quad\Longrightarrow\quad 1+h>1 \quad\Longrightarrow\quad f(1+h)=1+h.$$

Therefore, the limit in (**??**) is

$$\lim_{h\to 0+}\frac{1+h-1}{h}=\lim_{h\to 0-}1=1.$$

Since

$$2=\lim_{h\to 0-}\frac{f(1+h)-f(1)}{h}\neq\lim_{h\to 0+}\frac{f(1+h)-f(1)}{h}=1$$

f is not differentiable at $x=1$ (Definition 3.1).

G.5. $\boxed{\textit{Relevant page(s) in the notes: } 30, 31}$

Note that for any $x, h \in \mathbb{R}$ such that each of $x+1$ and $x+h+1$ are non-zero,

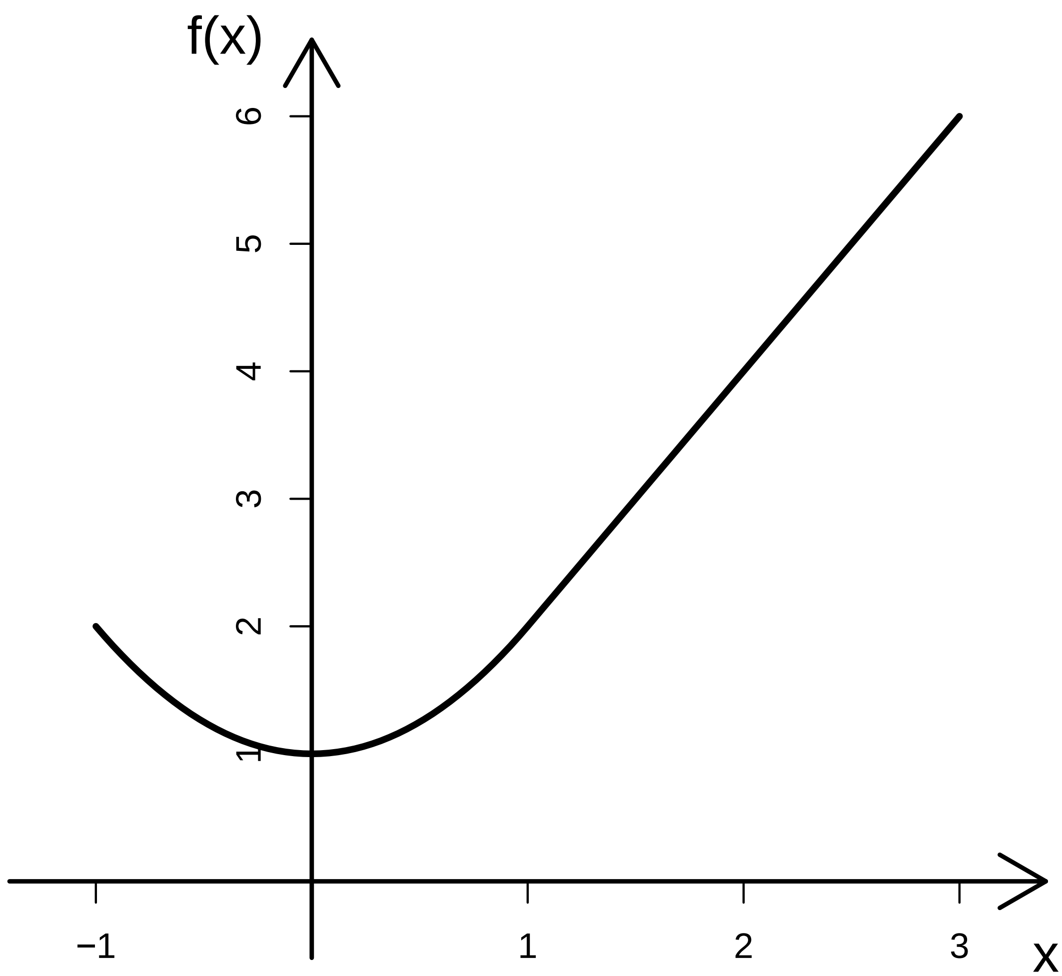
$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{1}{h}\left(\frac{x+h}{x+h+1} - \frac{x}{x+1}\right) \\ &= \frac{1}{(x+h+1)(x+1)}.\end{aligned}$$

Therefore, for all $x \neq -1$,

$$f'(x) = \lim_{h\to 0} \frac{1}{(x+h+1)(x+1)} = \frac{1}{(x+1)^2}.$$

G.6. $\boxed{\textit{Relevant page(s) in the notes: } 25, 30, 31}$

(a) The graph of f over the interval $[-1, 3]$ is shown in the accompanying figure.



(b) The derivative of f at $x = 1$ based on the left limit is

$$\lim_{h\to 0-} \frac{f(1+h)-f(1)}{h}. \tag{7}$$

The limit in (**??**) involves h approaching 0 from the left, which is such that

$$h < 0 \quad\implies\quad 1+h < 1 \quad\implies\quad f(1+h) = (1+h)^2 + 1.$$

Therefore, the limit in (**??**) is

$$\begin{aligned}\lim_{h\to 0-}\frac{(1+h)^2+1-2}{h} &= \lim_{h\to 0-}\frac{(1+h)^2+1-2}{h}\\ &= \lim_{h\to 0-}\frac{1+2h+h^2+1-2}{h}\\ &= \lim_{h\to 0-}(2+h)\\ &= 2.\end{aligned}$$

The derivative of f at $x=1$ based on the right limit is

$$\lim_{h\to 0+}\frac{f(1+h)-f(1)}{h}. \tag{8}$$

The limit in (**??**) involves h approaching 0 from the right, which is such that

$$h>0 \quad\Longrightarrow\quad 1+h>1 \quad\Longrightarrow\quad f(1+h)=2(1+h).$$

Therefore, the limit in (**??**) is

$$\lim_{h\to 0+}\frac{2(1+h)-2}{h}=\lim_{h\to 0-}2=2.$$

Since

$$\lim_{h\to 0-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0+}\frac{f(1+h)-f(1)}{h}=2$$

f is differentiable at $x=1$ and $f'(1)=2$ (Definition 3.1).

G.7. *Relevant page(s) in the notes:* 25, 30, 31

First note that for all $h>0$

$$\frac{f(0+h)-f(0)}{h}=\frac{|h|-|0|}{h}=\frac{h}{h}=1.$$

For any $\varepsilon>0$ let $\delta=1$ and note that,

$$0<h<\delta \quad\Longrightarrow\quad h>0 \quad\Longrightarrow\quad \frac{f(0+h)-f(0)}{h}=1 \quad\Longrightarrow\quad \left|\frac{f(0+h)-f(0)}{h}-1\right|<\varepsilon.$$

Therefore, by Definition 2.3,

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=1.$$

Next note that for all $h<0$

$$\frac{f(0+h)-f(0)}{h}=\frac{|h|-|0|}{h}=\frac{-h}{h}=-1.$$

For any $\varepsilon>0$ let $\delta=1$ and note that

$$-\delta<h<0 \quad\Longrightarrow\quad h<0 \quad\Longrightarrow\quad \frac{f(0+h)-f(0)}{h}=-1 \quad\Longrightarrow\quad \left|\frac{f(0+h)-f(0)}{h}-(-1)\right|<\varepsilon.$$

Therefore, by Definition 2.3,

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = -1.$$

From Proposition 2.4, the fact that

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} \neq \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h}$$

implies that

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \quad \text{does not exist.}$$

So by Definition 3.1 $f'(0)$ does not exist. In other words, $f(x) = |x|$ is not differentiable at $x = 0$.

G.8. *Relevant page(s) in the notes:* 30, 31, 32

Using the first identity given in the exercise note that for any $x, h \in \mathbb{R}$

$$\begin{aligned}
\frac{\sinh(x+h)-\sinh(x)}{h} &= \frac{\sinh(x)\cosh(h)+\cosh(x)\sinh(h)-\sinh(x)}{h} \\
&= \frac{\sinh(x)\{\cosh(h)-1\}+\cosh(x)\sinh(h)}{h} \\
&= \sinh(x)\left(\frac{e^h+e^{-h}-2}{2h}\right)+\cosh(x)\left(\frac{e^h-e^{-h}}{2h}\right) \\
&= \sinh(x)\left(\frac{e^h+e^{-h}-2}{2h}\right)+\cosh(x)\left(\frac{e^{2h}-1}{2h}\right)e^{-h} \\
&= \tfrac{1}{2}\sinh(x)\left\{\frac{e^{-h}(e^{2h}-2e^h+1)}{h}\right\}+\cosh(x)\left(\frac{e^{2h}-1}{2h}\right)e^{-h} \\
&= \tfrac{1}{2}\sinh(x)\left\{\frac{(e^h-1)}{h}\right\}^2 he^{-h}+\cosh(x)\left(\frac{e^{2h}-1}{2h}\right)e^{-h}.
\end{aligned}$$

Since, from the exercise question

$$\lim_{y\to 0} \frac{e^y-1}{y} = 1,$$

we have

$$\begin{aligned}
\lim_{h\to 0} \frac{\sinh(x+h)-\sinh(x)}{h} &= \tfrac{1}{2}\sinh(x)\left\{\lim_{h\to 0}\frac{(e^h-1)}{h}\right\}^2\left(\lim_{h\to 0} h\right)\left(\lim_{h\to 0} e^{-h}\right) \\
&\qquad + \cosh(x)\left\{\lim_{2h\to 0}\frac{(e^{2h}-1)}{2h}\right\}\left(\lim_{h\to 0} e^{-h}\right) \\
&= \tfrac{1}{2}\sinh(x)(1)^2(0)(1)+\cosh(x)(1)(1) \\
&= \cosh(x).
\end{aligned}$$

Using the second identity given in the exercise note that for any $x, h \in \mathbb{R}$

$$\begin{aligned}\frac{\cosh(x+h)-\cosh(x)}{h} &= \frac{\cosh(x)\cosh(h)+\sinh(x)\sinh(h)-\sinh(x)}{h}\\ &= \frac{\cosh(x)\{\cosh(h)-1\}+\sinh(x)\sinh(h)}{h}\end{aligned}$$

Using similar steps to those given for the derivative of $\sinh(x)$,

$$\lim_{h\to 0}\frac{\cosh(x+h)-\cosh(x)}{h} = \sinh(x).$$

In summary,

$$\sinh'(x) = \cosh(x) \quad \text{and} \quad \cosh'(x) = \sinh(x).$$

G.9. *Relevant page(s) in the notes:* 29, 37

Let

$$f(x) = x^7 + x^5 + x^3 + x + 1000.$$

Since f is a polynomial function of x, f is continuous and differentiable for all $x \in \mathbb{R}$. Also

$$f(-3) = -1460 \quad \text{and} \quad f(0) = 1000 \qquad \Longrightarrow f(-3)f(0) = -1460000 < 0.$$

Therefore, by Theorem 2.19, there is a real number $c \in [-3, 0]$ such that $f(c) = 0$. Hence, f has at least one root. Next note that

$$f'(x) = 7x^6 + 5x^4 + 3x^2 + 1 > 0 \quad \text{for all} \quad x \in \mathbb{R}.$$

To prove that c is the unique root suppose that is another root such that $d > c$. Then, by Rolle's Theorem (Theorem 3.9) there is a real number $r \in [c, d]$ such that $f'(r) = 0$. But this contradicts the above result that $f'(x) > 0$ for all $x \in \mathbb{R}$. A similar conclusion arises if d is another root such that $d < c$. Therefore c is the only root of f. In summary, the polynomial

$$x^7 + x^5 + x^3 + x + 1000$$

has exactly one root in $\mathbb{R}$.

G.10. *Relevant page(s) in the notes:* 37

The function $f(x) = ax^2 + bx + c$ is well-known to be differentiable over $\mathbb{R}$ with derivative

$$f'(x) = 2ax + b.$$

We will use proof by contradiction. Suppose that f has three roots r, s and t such that $r < s < t$. Then

$$f(r) = f(s) = 0$$

and by Rolle's Theorem (Theorem 3.9) there is $c_1 \in (r, s)$ such that

$$f'(c_1) = 0 \quad \Longleftrightarrow \quad 2ac_1 + b = 0 \quad \Longleftrightarrow \quad c_1 = -b/(2a).$$

In addition,

$$f(s) = f(t) = 0$$

and by Rolle's Theorem (Theorem 3.9) there is $c_2 \in (s, t)$ such that

$$f'(c_2) = 0 \quad \Longleftrightarrow \quad 2ac_1 + b = 0 \quad \Longleftrightarrow \quad c_2 = -b/(2a).$$

Therefore $-b/(2a) = c_1 = c_2$. But $c_1 = c_2$ is a contradiction of $c_1 \in (r, s)$ and $c_2 \in (s, t)$ so f cannot have three roots.

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