UNIVERSITY OF TECHNOLOGY SYDNEY

School of Mathematical and Physical Sciences

35007 Real Analysis

EXERCISE SET F – Solutions

THEME: Uniform and Lipschitz Continuity

Date of this version: **20th August 2026**[1]

F.1. *Relevant page(s) in the notes: 27*

This is publication-quality proof. See † at the end of this document for a full explanation.

First note that for any $x, y \in [3, 8]$ we have

$$6 \leq x + y \leq 16 \quad \Longrightarrow \quad |x + y| \leq 16. \tag{1}$$

For any given $\varepsilon > 0$ set $\delta = \varepsilon/16$. Then for each $x, y \in [3, 8]$,

$$|x - y| < \delta \quad \Longleftrightarrow \quad |x - y| < \varepsilon/16.$$

Combining this result with the last inequality in (1) results in

$$|x + y||x - y| < 16(\varepsilon/16) = \varepsilon.$$

This is equivalent to

$$|x^2 - y^2| < \varepsilon \quad \Longleftrightarrow \quad \big|f(x) - f(y)\big| < \varepsilon.$$

By Definition 2.13, $f : [3, 8] \to \mathbb{R}$ is uniformly continuous.

Preliminary background working that led to an effective δ choice

For each $\varepsilon > 0$ need to get to

$$|x^2 - y^2| < \varepsilon \quad \Longleftrightarrow \quad |x + y||x - y| < \varepsilon.$$

Since

$$x, y \in [3, 8] \quad \Longrightarrow \quad 6 \leq x + y \leq 16 \quad \Longrightarrow \quad |x + y| \leq 16$$

choosing $\delta = \varepsilon/16$ will allow the $|x - y| < \delta$ condition to be combined with the $|x + y| \leq 16$ condition and get us to $|x + y||x - y| < \varepsilon$.

F.2. *Relevant page(s) in the notes: 27*

[1] This document is less than a few months old which makes errors more likely. Updated versions with later dates may be released during the semester. Notifications will be sent via e-mail whenever an updated version is released.

Let $x, y \in [0, \infty)$ such that $x \geq y$. Then, noting the $(a+b)(a-b) = a^2 - b^2$ identity,

$$\begin{aligned}
\sqrt{x} - \sqrt{y} \leq \sqrt{x} - \sqrt{y} + 2\sqrt{y} \quad &\iff \quad \sqrt{x} - \sqrt{y} \leq \sqrt{x} + \sqrt{y} \\
&\implies (\sqrt{x} - \sqrt{y})^2 \leq (\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y}) \\
&\implies (\sqrt{x} - \sqrt{y})^2 \leq x - y \\
&\implies \sqrt{x} - \sqrt{y} \leq \sqrt{x - y} \\
&\iff |\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|}.
\end{aligned}$$

A summary of these just-completed steps is that for $x, y \in [0, \infty)$ such that $x \geq y$ we have

$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|}. \tag{2}$$

Given $\varepsilon > 0$ let $\delta = \varepsilon^2$. Then

$$|x - y| < \delta \quad \iff \quad |x - y| < \varepsilon^2 \quad \iff \quad \sqrt{|x - y|} < \varepsilon. \tag{3}$$

Combining the inequality in (2) with the last inequality in (3) we then obtain

$$|\sqrt{x} - \sqrt{y}| < \varepsilon \quad \iff \quad |f(x) - f(y)| < \varepsilon.$$

If $x, y \in [0, \infty)$ are such that $y \geq x$ then similar steps lead to the same conclusion. So for all $x, y \in [0, \infty)$

$$|x - y| < \delta \quad \implies \quad |f(x) - f(y)| < \varepsilon.$$

By Definition 2.13, $f : [0, \infty] \to \mathbb{R}$ is uniformly continuous.

F.3. *Relevant page(s) in the notes:* 27

For arbitrary $\delta > 0$ let

$$x = \frac{1}{\delta} \quad \text{and} \quad y = \frac{1}{\delta} + \frac{\delta}{2}.$$

Then $x, y \in [0, \infty)$ and

$$|x - y| = \left| \frac{1}{\delta} - \left(\frac{1}{\delta} + \frac{\delta}{2} \right) \right| = \frac{\delta}{2} < \delta.$$

However, noting the $a^2 - b^2 = (a-b)(a+b)$ identity,

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y||x + y| = \frac{\delta}{2} \left(\frac{2}{\delta} + \frac{\delta}{2} \right) = 1 + \frac{\delta^2}{4} > 1.$$

Therefore, if $\varepsilon = 1$ then we have $x, y \in [0, \infty)$ such that

$$|x - y| < \delta \quad \text{but} \quad |f(x) - f(y)| > \varepsilon.$$

By Definition 2.13 $f(x) = x^2$ is not uniformly continuous on $[0, \infty)$.

F.4. Relevant page(s) in the notes: 27

This is publication-quality proof. See † at the end of this document for a full explanation.

For each $\delta > 0$ let

$$x = \min(\delta, 1) \quad \text{and} \quad y = \frac{x}{2}. \tag{4}$$

Then $x, y \in (0, 5]$ and

$$|x - y| = \frac{x}{2} < x \leq \delta.$$

since the first inequality in (4) implies that $x \leq \delta$. However

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{1}{x} - \frac{2}{x}\right| = \frac{1}{x}. \tag{5}$$

From the first inequality in (4)

$$0 < x \leq 1 \quad \Longrightarrow \quad \frac{1}{x} \geq 1. \tag{6}$$

Combining (5) and (6) leads to

$$|f(x) - f(y)| \geq 1.$$

In summary, for any $\delta > 0$ we can can find $x, y \in (0, 5]$ such that

$$|x - y| < \delta \quad \text{but} \quad |f(x) - f(y)| \geq 1.$$

Hence, by Definition 2.13 with $\varepsilon = 1$, $f : (0, 5] \to \mathbb{R}$ is not uniformly continuous.

Preliminary background working that led to effective x and y choices

We only need one (x, y) pair in $(0, 5]$ to be such that

$$|x - y| < \delta \quad \textit{but} \quad |f(x) - f(y)| \geq 1.$$

So for $x \in (0, 5]$ we are free to choose $y \in (0, 5]$ as we please so to reduce the number of variables we will try $y = x/2$. With this choice

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{2}{x}\right| = \frac{1}{x}.$$

and so the condition $|f(x) - f(y)| \geq 1$ reduces to $1/x \geq 1$, which is equivalent to $x \leq 1$. The condition

$$|x - y| < \delta \quad \Longleftrightarrow \quad \frac{x}{2} < \delta \quad \textit{which is implied by the simpler} \quad x < \delta.$$

So we want both $x \leq 1$ and $x < \delta$. Setting $x = \min(\delta, 1)$ guarantees these.

F.5. Relevant page(s) in the notes: 6, 28

First note the algebraic identities

$$(a^2 - b^2) = (a + b)(a - b) \quad \text{and} \quad (a^3 - b^3) = (a^2 + ab + b^2)(a - b). \tag{7}$$

For any $x, y \in [2,5]$, using the triangle inequality and (7), we have

$$\begin{aligned}|f(x) - f(y)| &= |4(x^3 - y^3) - 7(x^2 - y^2)| \\ &\leq 4|x^3 - y^3| + 7|x^2 - y^2| \\ &= \left\{4|x^2 + xy + y^2| + 7|x + y|\right\}|x - y| \\ &\leq \left\{4|5^2 + 5 \times 5 + 5^2| + 7|5 + 5|\right\}|x - y| \\ &= 370|x - y|.\end{aligned}$$

Therefore, by Definition 2.17 with $M = 370$, $f : [2,5] \to \mathbb{R}$ is Lipschitz continuous.

F.6. *Relevant page(s) in the notes:* 28, 38

Let $x, y \in [2,5]$ be such that $x < y$. Then from the Mean Value Theorem (Theorem 3.10) there exits $c \in [x, y] \subseteq [2,5]$ such that

$$f(y) - f(x) = (y - x)f'(c).$$

Hence for all $x < y$ in $[2,5]$

$$|f(x) - f(y)| \leq |x - y| \sup\{|f'(c)| : c \in [2,5]\}. \tag{8}$$

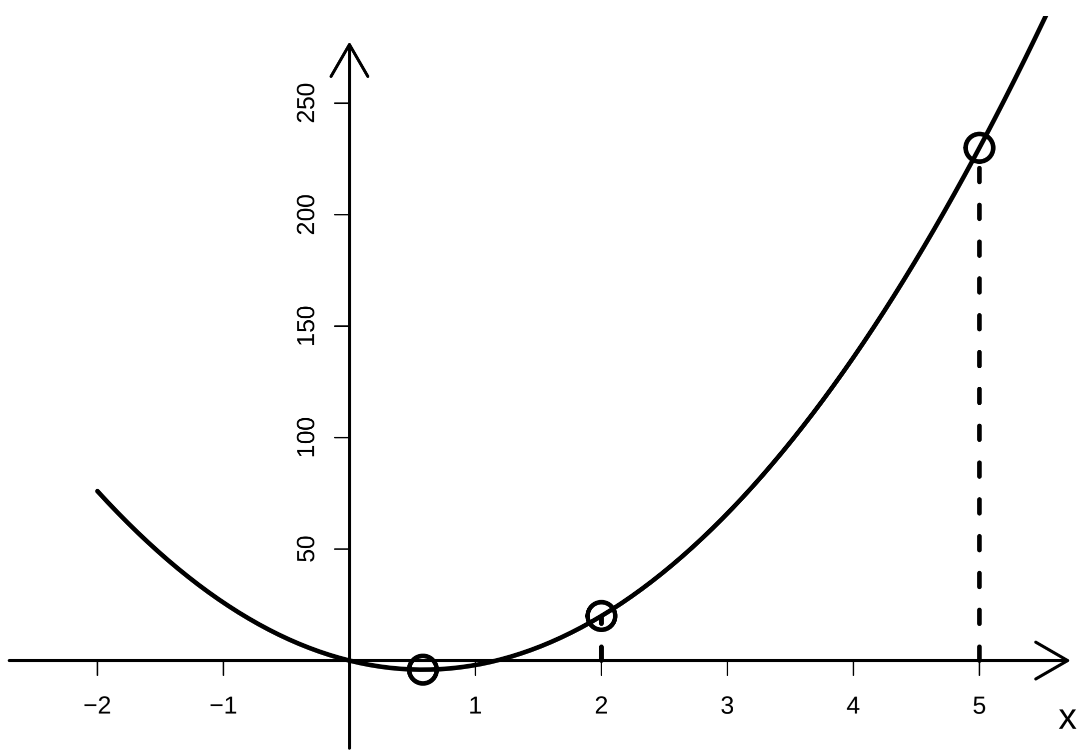
Similarly, for $y < x$ in $[2,5]$ condition (8) also holds. For $x = y$ in $[2,5]$ we have

$$0 = |f(x) - f(y)| = |x - y||M| = 0 \quad \text{for all } M \in \mathbb{R}.$$

Therefore (8) holds for all $x, y \in [2,5]$. Note that

$$f'(x) = 12x^2 - 14x$$

which corresponds to a U-shaped parabola with vertex at $x = -(-14)/(24) = 7/12$, as shown in the accompanying figure.



Since

$$f'(7/12) = -49/12, \quad f'(2) = 20 \quad \text{and} \quad f'(5) = 230$$

we then have

$$\sup\{|f'(c)| : c \in [2,5]\} = 230.$$

Therefore, for all $x < y$ in $[2,5]$

$$|f(x) - f(y)| \leq 230|x-y|$$

and by Definition 2.17 $f : [2,5] \to \mathbb{R}$ is Lipschitz continuous.

F.7. *Relevant page(s) in the notes:* 28

For any $x, y \in [-8,6]$ we have

$$|f(x) - f(y)| = \left|\frac{1}{x+11} - \frac{1}{y+11}\right| = \frac{|x-y|}{|x+11||y+11|}.$$

Now note that for all $x \in [-8,6]$

$$x \geq -8 \quad \Longrightarrow \quad x + 11 \geq -8 + 11 = 3 \quad \Longrightarrow \quad \frac{1}{|x+11|} \leq \frac{1}{3}.$$

Similarly, for all $y \in [-8,6]$,

$$\frac{1}{|y+11|} < \frac{1}{3}.$$

Therefore, for any $x, y \in [-8,6]$ we have

$$|f(x) - f(y)| \leq |x-y|/9.$$

Therefore, by Definition 2.17 with $M = 1/9$, $f : [-8,6] \to \mathbb{R}$ is Lipschitz continuous.

F.8. *Relevant page(s) in the notes:* 6, 28, 38

The first derivative of f is

$$f'(x) = \frac{d}{dx}\frac{e^x}{(1+e^x)^2} = \frac{e^x(1-e^x)}{(e^x+1)^3}.$$

Therefore

$$|f'(x)| = \frac{e^x}{1+e^x} \times \frac{1}{1+e^x} \times \frac{|e^x - 1|}{e^x+1} = \frac{1}{1+e^{-x}} \times \frac{1}{1+e^x} \times \frac{|e^x-1|}{e^x+1}.$$

Noting that

$$1 + e^{-x} > 1 \quad \Longrightarrow \quad \frac{1}{1+e^{-x}} < 1 \quad \text{and} \quad 1 + e^x > 1 \quad \Longrightarrow \quad \frac{1}{1+e^x} < 1$$

we then have

$$|f'(x)| < \frac{|e^x - 1|}{e^x + 1} \quad \text{for all } x \in \mathbb{R}.$$

Also, the triangle inequality gives

$$|e^x - 1| = |e^x + (-1)| \leq e^x + |-1| = e^x + 1 \qquad \Longrightarrow \qquad \frac{|e^x - 1|}{e^x + 1} \leq 1$$

so we now have

$$|f'(x)| \leq 1 \quad \text{for all } x \in \mathbb{R}. \tag{9}$$

Let $x, y \in \mathbb{R}$ be such that $x < y$. Then from the Mean Value Theorem (Theorem 3.10) there exits $c \in [x, y]$ such that

$$f(y) - f(x) = (y - x)f'(c)$$

Hence for all $x < y$

$$|f(x) - f(y)| \leq |x - y| \sup\{|f'(c)| : c \in \mathbb{R}\}.$$

Since, from (9),

$$\sup\{|f'(c)| : c \in \mathbb{R}\} \leq 1 \qquad \Longrightarrow \qquad |f(x) - f(y)| \leq |x - y| \quad \text{for all } x < y.$$

Similar arguments lead to

$$|f(x) - f(y)| \leq |x - y| \quad \text{for all } x > y.$$

Trivially, $|f(x) - f(y)| = |x - y|$ for $x = y$. Combining each of these findings we obtain

$$|f(x) - f(y)| \leq |x - y| \quad \text{for all } x, y \in \mathbb{R}$$

and by Definition 2.17 f is Lipschitz continuous over $\mathbb{R}$.

F.9. *Relevant page(s) in the notes:* 6, 28, 38

The first derivative of f is

$$f'(x) = \frac{-(x^6 + 6x^4 + 4x^3 - 30x^2 - 20)}{(x^4 + 10)^2}.$$

We seek a bound for $|f'(x)|$ over $x \in \mathbb{R}$ and treat the following cases separately:

(i) $|x| < 1$ and (ii) $|x| \geq 1$.

Case (i). Throughout Case (i) we have $|x| < 1$ which, with the assistance of the triangle inequality, leads to

$$\begin{aligned} |-(x^6 + 6x^4 + 4x^3 - 30x^2 - 20)| &\leq |x|^6 + 6|x|^4 + 4|x|^3 + 30|x|^2 + 20 \\ &\leq |x| + 6|x| + 4|x| + 30|x| + 20 \\ &< 61. \end{aligned}$$

Also

$$(x^4 + 10)^2 = x^8 + 20x^4 + 100 \geq 100 \qquad \Longrightarrow \qquad \frac{1}{(x^4 + 10)^2} \leq \frac{1}{100}.$$

Therefore, for Case (i), we have

$$|f'(x)| < \frac{61}{100} \quad \text{when } |x| < 1. \tag{10}$$

Case (ii). Throughout Case (ii) we have $|x| \geq 1$ which means that

$$1/|x| \leq 1. \tag{11}$$

We also have

$$(x^4+10)^2 = x^8 + 20x^4 + 100 \geq x^8 \quad \Longrightarrow \quad \frac{1}{(x^4+10)^2} \leq \frac{1}{x^8}.$$

and so, using (11),

$$\begin{aligned} |f'(x)| &= (|x|^6 + 6|x|^4 + 4|x|^3 + 30|x|^2 + 20)/|x|^8 \\ &= (1/|x|)^2 + 6(1/|x|)^4 + 4(1/|x|)^5 + 30(1/|x|)^6 + 20(1/|x|)^8 \end{aligned}$$

which leads to

$$|f'(x)| < 61 \quad \text{when } |x| \geq 1. \tag{12}$$

Combining (10) and (12) we have

$$|f'(x)| \leq 61 \quad \text{for all} \quad x \in \mathbb{R} \quad \Longrightarrow \quad \sup\{|f'(c)| : c \in \mathbb{R}\} \leq 61.$$

Let $x, y \in \mathbb{R}$ be such that $x < y$. Then from the Mean Value Theorem (Theorem 3.10) there exits $c \in [x, y]$ such that

$$f(y) - f(x) = (y - x)f'(c)$$

Hence for all $x < y$

$$|f(x) - f(y)| \leq |x - y| \sup\{|f'(c)| : c \in \mathbb{R}\}.$$

Since, from (12),

$$\sup\{|f'(c)| : c \in \mathbb{R}\} \leq 61 \quad \Longrightarrow \quad |f(x) - f(y)| \leq 61|x - y| \quad \text{for all } x < y.$$

Similar arguments lead to

$$|f(x) - f(y)| \leq 61|x - y| \quad \text{for all } x > y.$$

Trivially, $|f(x) - f(y)| = 61|x - y|$ for $x = y$. Combining each of these findings we obtain

$$|f(x) - f(y)| \leq 61|x - y| \quad \text{for all } x, y \in \mathbb{R}$$

and by Definition 2.17 f is Lipschitz continuous over $\mathbb{R}$.

F.10. *Relevant page(s) in the notes:* 27, 28

(a) Let $x, y \in [0,1]$ such that $x \geq y$. Then, noting the $(a+b)(a-b) = a^2 - b^2$ identity,

$$\begin{aligned}
\sqrt{x} - \sqrt{y} \leq \sqrt{x} - \sqrt{y} + 2\sqrt{y} \quad &\iff \quad \sqrt{x} - \sqrt{y} \leq \sqrt{x} + \sqrt{y} \\
&\implies (\sqrt{x} - \sqrt{y})^2 \leq (\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y}) \\
&\implies (\sqrt{x} - \sqrt{y})^2 \leq x - y \\
&\implies \sqrt{x} - \sqrt{y} \leq \sqrt{x - y} \\
&\iff |\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|}.
\end{aligned}$$

A summary of these just-completed steps is that for $x, y \in [0,1]$ such that $x \geq y$ we have

$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|}. \tag{13}$$

Given $\varepsilon > 0$ let $\delta = \varepsilon^2$. Then

$$|x - y| < \delta \quad \iff \quad |x - y| < \varepsilon^2 \quad \iff \quad \sqrt{|x - y|} < \varepsilon. \tag{14}$$

Combining the inequality in (13) with the last inequality in (14) we then obtain

$$|\sqrt{x} - \sqrt{y}| < \varepsilon \quad \iff \quad |f(x) - f(y)| < \varepsilon.$$

If $x, y \in [0,1]$ are such that $y \geq x$ then similar steps lead to the same conclusion. So for all $x, y \in [0,1]$

$$|x - y| < \delta \quad \implies \quad |f(x) - f(y)| < \varepsilon$$

By Definition 2.13, $f : [0,1] \to \mathbb{R}$ is uniformly continuous.

(b) We will use proof by contradiction. If $f : [0,1] \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ then, by Definition 2.17, there exists $M \geq 0$ such that

$$\text{for all } x, y \in [0,1], \quad |\sqrt{x} - \sqrt{y}| \leq M|x - y|.$$

Therefore, setting $y = 0$, we must have for all $0 < x \leq 1$:

$$\sqrt{x} \leq Mx \quad \iff \quad \frac{1}{\sqrt{x}} \leq M. \tag{15}$$

Now set $x = 1/(M+1)^2 \in (0,1]$ which implies that

$$\frac{1}{x} = (M+1)^2 \quad \iff \quad \frac{1}{\sqrt{x}} = M + 1$$

which is a contradiction of the second inequality of (15). Therefore Definition 2.17 cannot hold and $f : [0,1] \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ is not Lipschitz continuous.

† According to the lecturer's experiences in publishing proofs in mathematics and statistics journals over many years, preliminary background working **should not be included** in the proof. This is the convention in esteemed journals such as *The Annals of Mathematics* and *The Annals of Statistics*. Many web-sites and some books **do not follow** publication-quality conventions and tend to include such preliminary background working. Subject participants are encouraged to learn about and use publication-quality versions of mathematical proofs.

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