UNIVERSITY OF TECHNOLOGY SYDNEY

School of Mathematical and Physical Sciences

35007 Real Analysis

EXERCISE SET E – Solutions

THEME: **Limits and Continuity Theory**

Date of this version: **22nd July 2026**[1]

E.1. *Relevant page(s) in the notes:* 6, 24

This is publication-quality proof. See † at the end of this document for a full explanation.

First note that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is such that all elements of the domain of f are limit points. This is because, for any $x \in \mathbb{R}$, the trivial sequence $x_n = x,\ n \in \mathbb{N}$ converges to x. Therefore (1) of Definition 2.1 is satisfied.

For any given $\varepsilon > 0$ set $\delta = \min\big(1, \varepsilon/13\big)$. Then $|x-6| < \delta$ implies that

$$|x-6| < \frac{\varepsilon}{13} \quad \text{and} \quad |x-6| < 1. \tag{1}$$

Then, using the first inequality in (1) and the triangle inequality

$$|x^2-36| = |x^2-6^2| = |x+6||x-6| < \frac{\varepsilon|x+6|}{13} = \frac{\varepsilon|x-6+12|}{13} \le \frac{\varepsilon(|x-6|+12)}{13}.$$

Combining this result with the second inequality of (1) we obtain

$$|x^2-36| < \frac{\varepsilon(1+12)}{13} = \varepsilon.$$

Hence, by the ε-δ definition of a limit ((2) of Definition 2.1), $\lim\limits_{x\to 6} x^2 = 36$.

Preliminary background working that led to an effective δ choice

For any given $\varepsilon > 0$ we need to get to

$$|x^2-36| < \varepsilon \quad \iff \quad |x-6||x+6| < \varepsilon. \tag{2}$$

We are allowed to restrict $\delta > 0$ as we please so we will start by forcing it to be such that $0 < \delta \le 1$. Under this restriction, the condition

$$|x-6| < \delta \quad \implies \quad |x-6| < 1 \tag{3}$$

so from now on we are only considering x that are within a 1 unit radius centred on 6. Looking again at the second part of (2) we need to take care of the $|x+6|$ factor. By the triangle inequality

$$|x+6| = |x-6+12| < |x-6| + 12 < 13 \quad \textit{under the restriction in } (3).$$

[1] This document is less than a few months old which makes errors more likely. Updated versions with later dates may be released during the semester. Notifications will be sent via e-mail whenever an updated version is released.

Given this $|x+6| < 13$ bound and the

$$|x-6||x+6| < \varepsilon$$

requirement we need to add the extra restriction $|x-6| < \varepsilon/13$ so that we can have

$$|x-6||x+6| < (\varepsilon/13)13 = \varepsilon.$$

So for any $\varepsilon > 0$ we need both

- *x to be within a radius of length 1 centred on 6,*
- *x to be within a radius of length $\varepsilon/13$ centred on 6.*

Setting δ to be the smaller of 1 and $\varepsilon/13$ will guarantee both of these conditions. This leads to the choice

$$\delta = \min(1, \varepsilon/13).$$

E.2. *Relevant page(s) in the notes:* 24

This is publication-quality proof. See † at the end of this document for a full explanation.

First note that the function $f : \{x \in \mathbb{R} : x \neq 0\} \to \mathbb{R}$ given by $f(x) = 1/x$ is such that all elements of the domain of f are limit points. This is because, for any real number $x \neq 0$, the trivial sequence $x_n = x,\ n \in \mathbb{N}$ converges to x. Therefore (1) of Definition 2.1 is satisfied.

For any given $\varepsilon > 0$ set $\delta = \min\big(7/2, 49\varepsilon/2\big)$. Then $|x-7| < \delta$ implies that

$$|x-7| < 7/2 \quad \text{and} \quad |x-7| < 49\varepsilon/2. \tag{4}$$

The first inequality in (4) implies

$$-7/2 < x-7 < 7/2 \implies 7/2 < x < 21/2 \implies |x| > 7/2 \implies \frac{1}{|x|} < 2/7. \tag{5}$$

Using the last inequality in (5) and then the second inequality in (4) we then have

$$\left|\frac{1}{x} - \frac{1}{7}\right| = \frac{|x-7|}{7|x|} < \frac{2|x-7|}{49} < (2/49)(49\varepsilon/2) = \varepsilon.$$

Hence, by the ε-δ definition of a limit ((2) of Definition 2.1),

$$\lim_{x\to 7} \frac{1}{x} = \frac{1}{7}.$$

Preliminary background working that led to an effective δ choice

For any given $\varepsilon > 0$ we need to get to

$$\left|\frac{1}{x} - \frac{1}{7}\right| < \varepsilon \iff \frac{1}{7|x|} \times |x-7| < \varepsilon. \tag{6}$$

We want to get $|x|$ away from zero so that $1/|x|$ is well-behaved. This can be down by imposing a restriction such as $\delta \leq 7/2$ since it implies

$$|x-7| < 7/2 \iff -7/2 < x-7 < 7/2 \iff 7/2 < x < 21/2. \tag{7}$$

From (7) we then have

$$x > 7/2 \implies |x| > 7/2 \implies \frac{1}{|x|} < 2/7 \implies \frac{1}{7|x|} < 2/49.$$

Staring at (6) for a little while it also becomes apparent that we can get to $\left|\frac{1}{x}-\frac{1}{7}\right| < \varepsilon$ by having the restriction $\delta \leq 49\varepsilon/2$ since it implies

$$|x-7| < 49\varepsilon/2 \quad \textit{which when combined with } \frac{1}{7|x|} < 2/49$$

leads to, from (6),

$$\left|\frac{1}{x}-\frac{1}{7}\right| < (2/49)(49\varepsilon/2) = \varepsilon.$$

So for any $\varepsilon > 0$ we need both

- *x to be within a radius of length $7/2$ centred on 7,*
- *x to be within a radius of length $49\varepsilon/2$ centred on 7.*

Setting δ to be the smaller of $7/2$ and $49\varepsilon/2$ will guarantee both of these conditions. This leads to the choice

$$\delta = \min(7/2, 49\varepsilon/2).$$

E.3. *Relevant page(s) in the notes: 6, 24*

This is publication-quality proof. See † at the end of this document for a full explanation.

First note that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is such that all elements of the domain of f are limit points. This is because, for any $x \in \mathbb{R}$, the trivial sequence $x_n = x,\ n \in \mathbb{N}$ converges to x. Therefore (1) of Definition 2.1 is satisfied.

For any given $\varepsilon > 0$ set $\delta = \min\big(1, \varepsilon/61\big)$. Then $|x-4| < \delta$ implies that

$$|x-4| < \frac{\varepsilon}{61} \quad \text{and} \quad |x-4| < 1. \tag{8}$$

Then, using the first inequality in (8) and the triangle inequality

$$\begin{aligned}|x^3-64| = |x^3-4^3| = |x^2+4x+16||x-4| &< \frac{\varepsilon|x^2+4x+16|}{61} \\ &= \frac{\varepsilon\big||x-4+4|^2+4(x-4+4)+16\big|}{61} \\ &< \frac{\varepsilon\big\{(|x-4|+4)^2+4(|x-4|+4)+16\big\}}{61}.\end{aligned}$$

Combining this result with the second inequality of (8) we obtain

$$|x^3-64| < \frac{\varepsilon\big\{(1+4)^2+4(1+4)+16\big\}}{61} = \varepsilon.$$

Hence, by the ε-δ definition of a limit ((2) of Definition 2.1), $\lim_{x\to 4} x^3 = 64$.

Preliminary background working that led to an effective δ choice

We need to be aware of the difference of two cubes identity:

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2).$$

which will be used in the next step. For any given $\varepsilon > 0$ we need to get to

$$|x^3 - 64| < \varepsilon \quad \iff \quad |x-4||x^2 + 4x + 16| < \varepsilon. \tag{9}$$

We are allowed to restrict $\delta > 0$ as we please so we will start by forcing it to be such that $0 < \delta \leq 1$. Under this restriction, the condition

$$|x-4| < \delta \quad \implies \quad |x-4| < 1 \tag{10}$$

so from now on we are only considering x that are within a 1 unit radius centred on 4. Looking again at the second part of (9) we need to take care of the $|x^2+4x+16|$ factor. By the triangle inequality

$$\begin{aligned} |x^2 + 4x + 16| &= |(x-4+4)^2 + 4(x-4+4) + 16| \\ &< (|x-4| + 4)^2 + 4(|x-4| + 4) + 16 \\ &< (1+4)^2 + 4(1+4) + 16 = 61 \\ &\qquad \textit{under the restriction in } (10). \end{aligned}$$

Given this $|x^2+4x+16| < 61$ bound and the

$$|x-4||x^2+4x+16| < \varepsilon$$

requirement we need to add the extra restriction $|x-4| < \varepsilon/61$ so that we can have

$$|x-4||x^2+4x+16| < (\varepsilon/61)61 = \varepsilon.$$

So for any $\varepsilon > 0$ we need both

- *x to be within a radius of length 1 centred on 4,*
- *x to be within a radius of length $\varepsilon/61$ centred on 4.*

Setting δ to be the smaller of 1 and $\varepsilon/61$ will guarantee both of these conditions. This leads to the choice

$$\delta = \min(1, \varepsilon/61).$$

E.4. *Relevant page(s) in the notes:* 24

This is publication-quality proof. See † at the end of this document for a full explanation.

First note that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 1/(x^2+17)$ is such that all elements of the domain of f are limit points. This is because, for any $x \in \mathbb{R}$, the trivial sequence $x_n = x,\ n \in \mathbb{N}$ converges to x. Therefore (1) of Definition 2.1 is satisfied.

For any given $\varepsilon > 0$ set $\delta = \min\left(1, 1166\,\varepsilon/5\right)$. Then $|x-7| < \delta$ implies that

$$|x-7| < 1 \quad \text{and} \quad |x-7| < \frac{1166\,\varepsilon}{5}. \tag{11}$$

Then, using the first inequality in (11),

$$-1 < x-7 < 1 \quad \Longrightarrow \quad 6 < x < 8 \quad \Longrightarrow \quad 13 < x+7 < 15 \quad \Longrightarrow \quad |x+7| < 15. \tag{12}$$

Also, again using the first inequality in (11),

$$-1 < x-7 < 1 \quad \Longrightarrow \quad x > 6 \quad \Longrightarrow \quad x^2+17 > 53 \quad \Longrightarrow \quad \frac{1}{x^2+17} < \frac{1}{53}. \tag{13}$$

Using the final inequalities in (12) and (13) and then the last inequality in (11) we then have

$$\begin{aligned} |f(x) - f(7)| &= \left| \frac{1}{x^2+17} - \frac{1}{66} \right| = \frac{|x^2 - 7^2|}{66(x^2+17)} = |x+7| \left(\frac{1}{x^2+17} \right) \frac{|x-7|}{66} \\ &< 15 \left(\frac{1}{53} \right) \frac{|x-7|}{66} = \frac{5|x-7|}{1166} < \left(\frac{5}{1166} \right) \left(\frac{1166\varepsilon}{5} \right) = \varepsilon. \end{aligned}$$

Hence, by the ε-δ definition of a limit ((2) of Definition 2.1),

$$\lim_{x \to 7} \frac{1}{x^2+17} = \frac{1}{66}.$$

Preliminary background working that led to an effective δ choice

For any given $\varepsilon > 0$ we need to get to

$$\left| \frac{1}{x^2+17} - \frac{1}{66} \right| < \varepsilon \quad \Longleftrightarrow \quad \frac{|x^2 - 7^2|}{66(x^2+17)} < \varepsilon \quad \Longleftrightarrow \quad \frac{|x+7||x-7|}{66(x^2+17)} < \varepsilon. \tag{14}$$

We are allowed to restrict $\delta > 0$ as we please so we will start by forcing it to be such that $0 < \delta \leq 1$. Under this restriction, the condition

$$|x-7| < \delta \quad \Longrightarrow \quad |x-7| < 1 \tag{15}$$

so from now on we are only considering x that are within a 1 unit radius centred on 7. To take care of $|x+7|$ and $1/(x^2+17)$ factors in (14) we note that restriction (14) implies

$$6 < x < 8 \quad \Longrightarrow \quad 14 < x+7 < 15 \quad \Longrightarrow \quad |x+7| < 15 \tag{16}$$

as well as

$$6 < x < 8 \quad \Longrightarrow \quad x^2 > 36 \quad \Longrightarrow \quad x^2+17 > 53 \quad \Longrightarrow \quad \frac{1}{x^2+17} < \frac{1}{53}. \tag{17}$$

On combining the last inequalities in each of (15), (16) and (17) we have $|x-7| < \delta \leq 1$ implying

$$\frac{|x+7|}{66(x^2+17)} < \frac{15}{(66)(53)} = \frac{5}{1166}.$$

Therefore, to achieve the last inequality of (14) we also want to have

$$|x-7| < \frac{1166\varepsilon}{5}$$

so that we can have

$$\frac{|x+7||x-7|}{66(x^2+17)} < \left(\frac{5}{1166}\right)\left(\frac{1166\varepsilon}{5}\right) = \varepsilon.$$

So for any $\varepsilon > 0$ we need both

- *x to be within a radius of length 1 centred on 7,*
- *x to be within a radius of length $1166\varepsilon/5$ centred on 7.*

Setting δ to be the smaller of 1 and $1166\varepsilon/5$ will guarantee both of these conditions. This leads to the choice

$$\delta = \min(1, 1166\varepsilon/5).$$

E.5. *Relevant page(s) in the notes:* 6, 17, 24, 48, 49

This is publication-quality proof. See † at the end of this document for a full explanation.

For any $\varepsilon > 0$ let $\delta = \varepsilon/(1+\varepsilon)$. Then

$$|x-0| < \delta \quad \iff \quad |x| < \varepsilon/(1+\varepsilon) \tag{18}$$

Since e^x has Taylor series expansion

$$e^x = 1 + x + x^2/2! + x^3/3! + x^4/4! + \ldots \quad \text{for all } x \in \mathbb{R}$$

we then have, with use of the triangle inequality and then (18)

$$\begin{aligned}
|e^x - 1| &= |x + x^2/2! + x^3/3! + x^4/4! + \ldots| \\
&\le |x| + |x|^2/2! + |x|^3/3! + |x|^4/4! + \ldots \\
&\le |x| + |x|^2 + |x|^3 + |x|^4 + \ldots \\
&< \varepsilon/(1+\varepsilon) + \{\varepsilon/(1+\varepsilon)\}^2 + \{\varepsilon/(1+\varepsilon)\}^3 + \{\varepsilon/(1+\varepsilon)\}^4 + \ldots \\
&= \varepsilon/(1+\varepsilon)\big[1 + \varepsilon/(1+\varepsilon) + \{\varepsilon/(1+\varepsilon)\}^2 + \{\varepsilon/(1+\varepsilon)\}^3 + \ldots\big]
\end{aligned}$$

Noting the geometric series result

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{for all } |r| < 1$$

and the fact that $|\varepsilon/(1+\varepsilon)| < 1$ for any $\varepsilon > 0$ we then have

$$|e^x - 1| < \{\varepsilon/(1+\varepsilon)\}/\{1 - \varepsilon/(1+\varepsilon)\} = \varepsilon$$

By Definition 2.1

$$\lim_{x\to 0} e^x = 1.$$

Preliminary background working that led to an effective δ choice

For each $\varepsilon > 0$ we need to get to

$$|e^x - 1| < \varepsilon.$$

The Taylor series result for e^x implies that

$$|e^x - 1| = |x + x^2/2! + x^3/3! + x^4/4! + \ldots|. \tag{19}$$

We can bound the right-side side of (19) using the triangle inequality and then $1/n! \leq 1$ for all $n \in \mathbb{N}$ to get

$$|e^x - 1| \leq |x| + |x|^2/2! + |x|^3/3! + |x|^4/4! + \ldots \leq |x| + |x|^2 + |x|^3 + |x|^4 + \ldots$$

The proof starts with $|x| < \delta$ which implies that

$$|x| + |x|^2 + |x|^3 + \ldots < \delta + \delta^2 + \delta^3 + \ldots < \delta(1 + \delta + \delta^2 + \ldots)$$

If we first insist on $0 < \delta < 1$ then geometric series results lead to

$$\delta(1 + \delta + \delta^2 + \ldots) = \delta/(1 - \delta)$$

So we can get to $|e^x - 1| < \varepsilon$ by combining all of the above inequalities and choosing δ to satisfy

$$\delta/(1 - \delta) = \varepsilon.$$

This equation leads to $\delta = \varepsilon/(1 + \varepsilon)$ which also happens to be between 0 and 1 for all $\varepsilon > 0$.

E.6. *Relevant page(s) in the notes:* 26

We show that the definition of f being continuous at x (Definition 2.7) fails when $x = 0$.

Note that $f(0) = 1$, $f(y) = 0$ for all $y < 0$ and let $\varepsilon = \frac{1}{2}$. Then for any any $\delta_x > 0$

$$|x - y| < \delta_x \iff |y| < \delta_x. \tag{20}$$

However, regardless of the choice of $\delta_x > 0$, the condition in (20) does not imply that

$$|f(x) - f(y)| < \tfrac{1}{2}$$

since, for example, $y = -\frac{1}{2}\delta_x$ satisfies (20) but

$$|f(0) - f(-\tfrac{1}{2}\delta_x)| = |1 - 0| = 1 > \tfrac{1}{2}.$$

Therefore, Definition 2.7 does not apply and f is discontinuous at $x = 0$.

E.7. *Relevant page(s) in the notes:* 6, 24

This is publication-quality proof. See † at the end of this document for a full explanation.

First note that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is such that all elements of the domain of f are limit points. This is because, for any $x \in \mathbb{R}$, the trivial sequence $x_n = x,\ n \in \mathbb{N}$ converges to x. Therefore (1) of Definition 2.1 is satisfied.

For any given $\varepsilon > 0$ set $\delta = \min\big(1, \varepsilon/(2|c| + 1)\big)$. Then $|x - c| < \delta$ implies that

$$|x - c| < \frac{\varepsilon}{2|c| + 1} \quad \text{and} \quad |x - c| < 1. \tag{21}$$

Then, using the first inequality in (21) and the triangle inequality

$$|x^2 - c^2| = |x-c||x+c| < \frac{\varepsilon|x+c|}{2|c|+1} = \frac{\varepsilon|x-c+2c|}{2|c|+1} \leq \frac{\varepsilon(|x-c|+2|c|)}{2|c|+1}.$$

Combining this result with the second inequality of (21) we obtain

$$|x^2 - c^2| < \frac{\varepsilon(1+2|c|)}{2|c|+1} = \varepsilon.$$

Hence, by the ε-δ definition of a limit ((2) of Definition 2.1), $\lim_{x \to c} x^2 = c^2$.

Preliminary background working that led to an effective δ choice

For any given $\varepsilon > 0$ we need to get to

$$|x^2 - c^2| < \varepsilon \quad \Longleftrightarrow \quad |x-c||x+c| < \varepsilon. \tag{22}$$

We are allowed to restrict $\delta > 0$ as we please so we will start by forcing it to be such that $0 < \delta \leq 1$. Under this restriction, the condition

$$|x-c| < \delta \quad \Longrightarrow \quad |x-c| < 1 \tag{23}$$

so from now on we are only considering x that are within a 1 unit radius centred on c. Looking again at the second part of (22) we need to take care of the $|x+c|$ factor. By the triangle inequality

$$|x+c| = |x-c+2c| < |x-c| + |c| < 2|c|+1 \quad \textit{under the restriction in } (23).$$

Given this $|x+c| < 2|c|+1$ bound and the

$$|x-c||x+c| < \varepsilon$$

requirement we need to add the extra restriction $|x-c| < \varepsilon/(2|c|+1)$ so that we can have

$$|x-c||x+c| < \{\varepsilon/(2|c|+1)\}(2|c|+1) = \varepsilon.$$

So for any $\varepsilon > 0$ we need both

- *x to be within a radius of length 1 centred on c,*
- *x to be within a radius of length $\varepsilon/(2|c|+1)$ centred on c.*

Setting δ to be the smaller of 1 and $\varepsilon/(2|c|+1)$ will guarantee both of these conditions. This leads to the choice

$$\delta = \min\big(1, \varepsilon/(2|c|+1)\big).$$

E.8. *Relevant page(s) in the notes:* 24

This is publication-quality proof. See † at the end of this document for a full explanation.

First note that the function $f : \{x \in \mathbb{R} : x \neq 0\} \to \mathbb{R}$ given by $f(x) = 1/x$ is such that all elements of the domain of f are limit points. This is because, for any real number

$x \neq 0$, the trivial sequence $x_n = x,\ n \in \mathbb{N}$ converges to x. Therefore (1) of Definition 2.1 is satisfied.

For any given $\varepsilon > 0$ set $\delta = \min\big(|c|/2, c^2\varepsilon/2\big)$. Then $|x-c| < \delta$ implies that

$$|x-c| < |c|/2 \quad \text{and} \quad |x-c| < c^2\varepsilon/2. \tag{24}$$

In the case where $c > 0$, the first inequality in (24) implies

$$\begin{aligned} -c/2 < x - c < c/2 \quad &\Longrightarrow \quad c/2 < x < 3c/2 \\ \Longrightarrow \quad x > c/2 \ \ (\text{with } x, c > 0) \ &\Longleftrightarrow |x| > |c|/2. \end{aligned} \tag{25}$$

In the case where $c < 0$, the first inequality in (24) implies

$$\begin{aligned} c/2 < x - c < -c/2 \quad \Longrightarrow \quad 3c/2 < x < c/2 \quad &\Longrightarrow \quad -c/2 < -x < -3c/2 \\ \Longrightarrow \quad (-x) > (-c)/2 \ \ (\text{with } x, c < 0) \ &\Longleftrightarrow |x| > |c|/2. \end{aligned} \tag{26}$$

On combining the conclusions in (25) and (26), the first inequality in (24) implies

$$|x| > |c|/2 \quad \text{implies} \quad 1/|x| < 2/|c|. \tag{27}$$

Using the last inequality in (27) and then the second inequality in (24) we then have

$$\left|\frac{1}{x} - \frac{1}{c}\right| = \frac{|x-c|}{|c||x|} < \frac{2|x-c|}{c^2} < (2/c^2)(c^2\varepsilon/2) = \varepsilon.$$

Hence, by the ε-δ definition of a limit ((2) of Definition 2.1),

$$\lim_{x\to c} \frac{1}{x} = \frac{1}{c}.$$

Preliminary background working that led to an effective δ choice

For any given $\varepsilon > 0$ we need to get to

$$\left|\frac{1}{x} - \frac{1}{c}\right| < \varepsilon \quad \Longleftrightarrow \quad \frac{1}{|c||x|} \times |x-c| < \varepsilon. \tag{28}$$

We want to get $|x|$ away from zero so that $1/|x|$ is well-behaved. This can be down by imposing a restriction such as $\delta \le |c|/2$ since it implies

$$|x-c| < |c|/2 \quad \Longleftrightarrow \quad -|c|/2 < x - c < |c|/2 \quad \Longleftrightarrow \quad c - |c|/2 < x < c + |c|/2. \tag{29}$$

In the $c > 0$ case the last part of (29) gives $x > |c| - |c|/2 = |c|/2$ which implies $|x| > |c|/2$. In the $c < 0$ case the last part of (29) gives $-x > (-c) - |c|/2 = |c|/2$ which implies $|x| > |c|/2$. Therefore

$$\textit{the } \delta \le |c|/2 \textit{ restriction implies } |x| > |c|/2 \quad \textit{for all } c \neq 0. \tag{30}$$

From (30) we then have

$$|x| > |c|/2 \quad \Longrightarrow \quad \frac{1}{|x|} < \frac{2}{|c|} \quad \Longrightarrow \quad \frac{1}{|c||x|} < \frac{2}{c^2}.$$

Staring at (28) for a little while it also becomes apparent that we can get to $\left|\frac{1}{x} - \frac{1}{c}\right| < \varepsilon$ *by having the restriction* $\delta \le c^2\varepsilon/2$ *since it implies*

$$|x - c| < c^2\varepsilon/2 \quad \textit{which when combined with } \frac{1}{|c||x|} < \frac{2}{c^2}$$

leads to, from (28),

$$\left|\frac{1}{x} - \frac{1}{c}\right| < (2/c^2)(c^2\varepsilon/2) = \varepsilon.$$

So for any $\varepsilon > 0$ *we need both*

- x *to be within a radius of length* $|c|/2$ *centred on* c*,*
- x *to be within a radius of length* $c^2\varepsilon/2$ *centred on* c*.*

Setting δ *to be the smaller of* $|c|/2$ *and* $c^2\varepsilon/2$ *will guarantee both of these conditions. This leads to the choice*

$$\delta = \min(|c|/2, c^2\varepsilon/2).$$

E.9. *Relevant page(s) in the notes: 6, 24*

This is publication-quality proof. See † at the end of this document for a full explanation.

First note that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is such that all elements of the domain of f are limit points. This is because, for any $x \in \mathbb{R}$, the trivial sequence $x_n = x,\ n \in \mathbb{N}$ converges to x. Therefore (1) of Definition 2.1 is satisfied.

For any given $\varepsilon > 0$ set

$$\delta = \min\left(1, \frac{\varepsilon}{3c^2 + 3|c| + 1}\right).$$

Then $|x - c| < \delta$ implies that

$$|x - c| < \frac{\varepsilon}{3c^2 + 3|c| + 1} \quad \text{and} \quad |x - c| < 1. \tag{31}$$

Then, using the first inequality in (31) and the triangle inequality

$$\begin{aligned} |x^3 - c^3| = |x^3 - c^3| = |x^2 + cx + c^2||x - c| &< \frac{\varepsilon|x^2 + cx + c^2|}{3c^2 + 3|c| + 1} \\ &= \frac{\varepsilon\big||x - c + c|^2 + c(x - c + c) + c^2\big|}{3c^2 + 3|c| + 1} \\ &< \frac{\varepsilon\big\{(|x - c| + |c|)^2 + |c|(|x - c| + |c|) + c^2\big\}}{3c^2 + 3|c| + 1}. \end{aligned}$$

Combining this result with the second inequality of (31) we obtain

$$|x^3 - c^3| < \frac{\varepsilon\big\{(1 + |c|)^2 + |c|(1 + |c|) + c^2\big\}}{3c^2 + 3|c| + 1} = \varepsilon.$$

Hence, by the ε-δ definition of a limit ((2) of Definition 2.1), $\lim\limits_{x \to c} x^3 = c^3$.

Preliminary background working that led to an effective δ choice

We need to be aware of the difference of two cubes identity:

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2).$$

which will be used in the next step. For any given $\varepsilon > 0$ we need to get to

$$|x^3 - c^3| < \varepsilon \quad \iff \quad |x-c||x^2 + xc + c^2| < \varepsilon. \tag{32}$$

We are allowed to restrict $\delta > 0$ as we please so we will start by forcing it to be such that $0 < \delta \leq 1$. Under this restriction, the condition

$$|x-c| < \delta \quad \implies \quad |x-c| < 1 \tag{33}$$

so from now on we are only considering x that are within a 1 unit radius centred on c. Looking again at the second part of (32) we need to take care of the $|x^2 + cx + c^2|$ factor. By the triangle inequality

$$\begin{aligned}|x^2 + cx + c^2| &= |(x-c+c)^2 + c(x-c+c) + c^2| \\ &< (|x-c| + |c|)^2 + |c|(|x-c| + |c|) + c^2 \\ &< (1+|c|)^2 + |c|(1+|c|) + c^2 \\ &= 3c^2 + 3|c| + 1 \\ &\qquad \textit{under the restriction in } (33).\end{aligned}$$

Given this $|c^2 + cx + c^2| < 3c^2 + 3|c| + 1$ bound and the

$$|x-c||x^2 + cx + c^2| < \varepsilon$$

requirement we need to add the extra restriction

$$|x-c| < \varepsilon/(3c^2 + 3|c| + 1)$$

so that we can have

$$|x-c||x^2 + cx + c^2| < \left\{\varepsilon/\left(3c^2 + 3|c| + 1\right)\right\}\left(3c^2 + 3|c| + 1\right) = \varepsilon.$$

So for any $\varepsilon > 0$ we need both

- *x to be within a radius of length 1 centred on c,*
- *x to be within a radius of length $\varepsilon/\left(3c^2 + 3|c| + 1\right)$ centred on c.*

Setting δ to be the smaller of 1 and $\varepsilon/\left(3c^2 + 3|c| + 1\right)$ will guarantee both of these conditions. This leads to the choice

$$\delta = \min\left(1, \varepsilon/\left(3c^2 + 3|c| + 1\right)\right).$$

E.10. *Relevant page(s) in the notes: 26*

We show that the definition of f being continuous at x (Definition 2.7) fails when $x = 3$.

Note that $f(3) = 1$ and let $\varepsilon = \frac{1}{2}$. Then for any any $\delta_x > 0$ and $y \in \mathbb{R}$

$$|x - y| < \delta_x \iff |y - 3| < \delta_x \iff 3 - \delta_x < y < 3 + \delta_x. \tag{34}$$

However, regardless of the choice of $\delta_x > 0$, the condition in (34) does not imply that

$$|f(x) - f(y)| < \tfrac{1}{2}$$

since, for example,

$$y = \text{an irrational number in the interval } (3, 3 + \delta_x)$$

(which, based on the fact provided in the exercise, must exist) satisfies (34) but, since $f(y) = 0$,

$$|f(3) - f(y)| = |1 - 0| = 1 > \tfrac{1}{2}.$$

Therefore, Definition 2.7 does not apply and f is discontinuous at $x = 3$.

† According to the lecturer's experiences in publishing proofs in mathematics and statistics journals over many years, preliminary background working **should not be included** in the proof. This is the convention in esteemed journals such as *The Annals of Mathematics* and *The Annals of Statistics*. Many web-sites and some books **do not follow** publication-quality conventions and tend to include such preliminary background working. Subject participants are encouraged to learn about and use publication-quality versions of mathematical proofs.

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