UNIVERSITY OF TECHNOLOGY SYDNEY

School of Mathematical and Physical Sciences

35007 Real Analysis

EXERCISE SET D – Solutions

THEME: Series

Date of this version: **22nd July 2026**[1]

D.1. *Relevant page(s) in the notes:* 19

First note that
$$a_n \equiv \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} > 0 \quad \text{for all } n \in \mathbb{N}.$$
Define
$$b_n \equiv \frac{1}{n^2}$$
and note that
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n(n+1)} = \lim_{n\to\infty} \frac{1}{1+/n} = 1 \neq 0.$$
So by the Limit Comparison Test (Theorem 1.40) the series in the exercise converges.

D.2. *Relevant page(s) in the notes:* 22, Supplementary Results

(a) We have
$$s_n = 1 + r + r^2 + \ldots + r^n$$
$$\text{and} \quad rs_n = r + r^2 + r^3 + \ldots + r^{n+1}.$$
Subtraction of the first equation from the second equation leads to, for all $r \neq 1$,
$$(r-1)s_n = r^{n+1} - 1 \quad \Longrightarrow \quad s_n = \frac{r^{n+1}-1}{r-1}.$$

(b) The following cases are treated separately:

(i) $r = 0$, (ii) $r = 1$, (iii) $r = -1$, (iv) $r \neq 0$ and $|r| < 1$ and (v) $|r| > 1$.

When $r = 0$, $s_n = 1$ for all $n \in \mathbb{N}$ so, trivially $\lim_{n\to\infty} s_n = 1 = 1/(1-0)$. Hence, the result holds when $r = 0$.

When $r = 1$, $s_n = n + 1$ for all $n \in \mathbb{N}$ which is unbounded. Hence s_n diverges and the results holds when $r = 1$.

When $r = -1$, $s_n = \frac{1}{2}\{1 + (-1)^n\}$ for all $n \in \mathbb{N}$. Then, based on the n odd subsequence
$$\liminf_{n\to\infty} s_n = 0$$

[1]This document is less than a few months old which makes errors more likely. Updated versions with later dates may be released during the semester. Notifications will be sent via e-mail whenever an updated version is released.

and based on the n odd subsequence

$$\limsup_{n\to\infty} s_n = 1.$$

Since

$$0 = \liminf_{n\to\infty} s_n \neq \limsup_{n\to\infty} s_n = 1$$

s_n diverges (Proposition 1.44) and the result holds for $r = -1$.
When $r \neq 0$ and $r \neq 1$,

$$s_n = \frac{r^n - 1}{r - 1} \qquad \Longrightarrow \qquad \lim_{n\to\infty} s_n = \frac{\lim_{n\to\infty} a_n - 1}{r-1}. \quad \text{where} \quad a_n \equiv r^n.$$

Since

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|r|^{n+1}}{|r|^n} = |r|$$

and $|r| < 1$ by the Ratio Test for Sequences (Theorem S.1 of the Supplementary Results)

$$\lim_{n\to\infty} a_n = 0 \quad \text{and so} \quad \lim_{n\to\infty} s_n = \frac{0-1}{r-1} = \frac{1}{1-r}.$$

and the result holds for $|r| < 1$, $r \neq 0$.
When $|r| > 1$ by the Ratio Test for Sequences (Theorem S.1 of the Supplementary Results) a_n diverges, which implies that

$$s_n = \frac{a_n - 1}{r - 1}$$

diverges and the result holds for $|r| > 1$.

D.3. $\boxed{\textit{Relevant page(s) in the notes: } 18}$

For all $n \in \mathbb{N}$

$$4n^2 > n^2 \qquad \Longrightarrow \qquad 4n^2 + 15 > n^2 \qquad \Longrightarrow \qquad \frac{1}{4n^2+15} < \frac{1}{n^2}. \tag{1}$$

It is well-known (e.g. equation (1.2) in the notes) that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{is convergent.}$$

From the last inequality in (1) and the Comparison Test (Theorem 1.39)

$$\sum_{n=1}^{\infty} \frac{1}{4n^2+15} \quad \text{is convergent.}$$

D.4. $\boxed{\textit{Relevant page(s) in the notes: } 16, 19}$

The series of interest is

$$\sum_{n=1}^{\infty} a_n \quad \text{where} \quad a_n \equiv \frac{9+\sqrt{n}}{1+2n-n^2}$$

The a_n are not strictly positive for all $n \in \mathbb{N}$ so, for example, the Limit Comparison Test (Theorem 1.40) does not apply. However, note that

$$-\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{9+\sqrt{n}}{n^2-2n-1} = \sum_{n=1}^{2} \frac{9+\sqrt{n}}{n^2-2n-1} + \sum_{n=3}^{\infty} \frac{9+\sqrt{n}}{n^2-2n-1}. \tag{2}$$

The first term on the far right-hand side of (2) is finite and equal to $-(14+\sqrt{2})$. Making the substitution $m = n-2$ into the second term on the far right-hand side of (2) leads to it equalling

$$\sum_{m=1}^{\infty} \frac{9+\sqrt{m+2}}{(m+2)^2-2(m+2)-1} = \sum_{n=1}^{\infty} \widetilde{a}_n \quad \text{where} \quad \widetilde{a}_n \equiv \frac{9+\sqrt{n+2}}{(n+2)^2-2(n+2)-1}.$$

For $n \in \mathbb{N}$ let $b_n \equiv n^{-3/2}$ and note that (with division by n^2 on the numerator and denominator)

$$\frac{\widetilde{a}_n}{b_n} = \frac{(9+\sqrt{n+2})n^{3/2}}{\{(n+2)^2-2(n+2)-1\}} = \frac{(9/\sqrt{n}+\sqrt{1+2/n})}{\{(1+2/n)^2-2(1/n+2/n^2)-1/n^2\}}$$

which leads to

$$\lim_{n\to\infty} \frac{\widetilde{a}_n}{b_n} = 1 \neq 0.$$

So by the Limit Comparison Test (Theorem 1.40) $\sum_{n=1}^{\infty} \widetilde{a}_n$ converges. Hence, by Theorem 1.34,

$$\sum_{n=1}^{\infty} a_n = 14+\sqrt{2} - \sum_{n=1}^{\infty} \widetilde{a}_n \quad \text{also converges.}$$

D.5. *Relevant page(s) in the notes:* 15, 19

Define the sequences

$$a_n \equiv \frac{14n^5+3n^3}{9n^6+7n^2+5} \quad \text{and} \quad b_n \equiv \frac{1}{n}, \qquad n \in \mathbb{N}.$$

Then $\sum_{n=1}^{\infty} b_n$ is the harmonic series, which diverges. Next note that

$$\frac{a_n}{b_n} = \frac{14n^6+3n^4}{9n^6+7n^2+5} = \frac{14+3/n^2}{9+7/n^2+5/n^4}$$

which leads to

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{14+3/n^2}{9+7/n^2+5/n^4} = \frac{14}{9} \neq 0.$$

By the Limit Comparison Test (Theorem 1.40), the series in the exercise diverges.

D.6. *Relevant page(s) in the notes:* 23, 26

Define

$$a_n = \frac{1}{(11n+8)^{1/94}}, \qquad n \in \mathbb{N}$$

and note that, for all $n \in \mathbb{N}$,

$$11(n+1)+8 = 11n+19 > 11n+9. \tag{3}$$

Since the function $g(x) = x^{1/94}$ is monotone increasing over $\{x : x > 0\}$, (3) implies that

$$\{11(n+1)+8\}^{1/94} > (11n+8)^{1/94}$$

and therefore

$$\frac{1}{\{11(n+1)+8\}^{1/94}} < \frac{1}{(11n+8)^{1/94}} \quad \iff \quad a_{n+1} < a_n \quad \text{for all } n \in \mathbb{N}.$$

Also,

$$a_n = \left(\frac{1}{11n+8}\right)^{1/94} = \left\{\left(\frac{1}{n}\right)\left(\frac{1}{11+8/n}\right)\right\}^{1/94}$$

which, using the fact that g is continuous (Theorem 2.8), leads to

$$\lim_{n\to\infty} a_n = \left[\lim_{n\to\infty}\left\{\left(\frac{1}{n}\right)\left(\frac{1}{11+8/n}\right)\right\}\right]^{1/94} = 0.$$

By Theorem 1.48

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(11n+8)^{1/94}}$$

is convergent.

D.7. $\boxed{\textit{Relevant page(s) in the notes: } 18}$

First note that

$$\sum_{n=1}^{\infty} \frac{1}{n^2+6n+9} = \sum_{n=1}^{\infty} \frac{1}{(n+3)^2}.$$

Making the change of variables $m = n+3$ we then have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{(n+3)^2} &= \sum_{m=4}^{\infty} \frac{1}{m^2} \\
&= \sum_{m=1}^{\infty} \frac{1}{m^2} - \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}\right) \\
&= \frac{\pi^2}{6} - \left(1 + \frac{1}{4} + \frac{1}{9}\right) \\
&= \frac{6\pi^2 - 49}{36}.
\end{aligned}$$

D.8. $\boxed{\textit{Relevant page(s) in the notes: } 18}$

$\boxed{\textit{This is publication-quality proof. See † at the end of this document for a full explanation.}}$

First we check whether or not all terms in

$$\sum_{n=1}^{\infty} \frac{1}{5n^2 - 837} \tag{4}$$

are finite. The quadratic equation

$$5x^2 - 837 = 0 \quad \text{has roots} \quad x = \pm\sqrt{837/5}.$$

Since $873/5$ is not an integer it cannot be a perfect square and so neither of the roots is a natural number. Hence $5n^2 - 837 \neq 0$ for all $n \in \mathbb{N}$, which implies that all terms in (4) are finite. Let $N = 15$ and note that

$$n \geq N \quad \Longrightarrow \quad n^2 \geq 225 > 837/4 \quad \Longrightarrow \quad 4n^2 > 837 \quad \Longrightarrow \quad 5n^2 - 837 > n^2.$$

Therefore,

$$\text{for all} \quad n \geq 15 \quad \text{we have} \quad \frac{1}{5n^2 - 837} < \frac{1}{n^2}.$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

by the Comparison Test (Theorem 1.39)

$$\sum_{n=1}^{\infty} \frac{1}{5n^2 - 837} \quad \text{is convergent.}$$

Preliminary background working that led to an effective N choice

We know that

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent so it is anticipated that the same is true for

$$\sum_{n=1}^{\infty} \frac{1}{5n^2 - 837}.$$

To use the Comparison Test we want values of n for which

$$5n^2 - 837 \geq n^2 \quad \Longleftrightarrow \quad \frac{1}{5n^2 - 837} \leq \frac{1}{n^2}.$$

However,

$$5n^2 - 837 \geq n^2 \quad \Longleftrightarrow \quad n^2 \geq 837/4. \tag{5}$$

Therefore taking

$$n > \sqrt{837/4} = 14.465\ldots$$

will guarantee (5). It follows that $N = 15$ is sufficient.

D.9. $\boxed{\textit{Relevant page(s) in the notes: } 16, 18}$

Let a_n, $n \in \mathbb{N}$, be a sequence such that

$$\sum_{n=1}^{\infty} |a_n| \quad \text{converges.} \tag{6}$$

By definition, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Next let

$$b_n \equiv a_n + |a_n|, \quad n \in \mathbb{N}.$$

Since, for any $x \in \mathbb{R}$,

$$x + |x| = \begin{cases} x + (-x) = 0 & \text{when } x \leq 0, \\ x + x = 2x = 2|x| & \text{when } x > 0, \end{cases}$$

we have

$$0 \leq b_n \leq 2|a_n| \quad \text{for all } n \in \mathbb{N}.$$

Since, from (6), $\sum_{n=1}^{\infty} |a_n|$ is finite the same is true for $\sum_{n=1}^{\infty}(2|a_n|)$. So from the Comparison Test (Theorem 1.39)

$$\sum_{n=1}^{\infty} b_n \quad \text{converges.}$$

Observing that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} |a_n| \tag{7}$$

and the fact that both series on the right-hand side of converge, it follows from Theorem 1.34 that

$$\sum_{n=1}^{\infty} a_n \quad \text{converges.}$$

Thus, every absolutely convergent series is convergent.

D.10. $\boxed{\textit{Relevant page(s) in the notes: } 23, 26}$

Define

$$a_n \equiv \frac{1}{\ln\big(\ln(15n+28)\big)}, \quad n \in \mathbb{N}$$

and note that, for all $n \in \mathbb{N}$,

$$15(n+1) + 28 = 15n + 43 > 15n + 28. \tag{8}$$

Since the function $\ln(x)$ is monotone increasing over $\{x : x > 0\}$, (8) implies that

$$\ln\big(\ln\big(15(n+1)+28\big) > \ln\big(\ln(15n+28)\big)$$

Also,

$$\ln\big(\ln(15n+28)\big) \geq \ln\big(\ln(43)\big) > 0 \quad \text{for all } n \in \mathbb{N}.$$

Therefore,

$$\frac{1}{\ln\big(\ln\big(15(n+1)+28\big)} < \frac{1}{\ln\big(\ln(15n+28)\big)} \quad \Longleftrightarrow \quad a_{n+1} < a_n \quad \text{for all } n \in \mathbb{N}.$$

Also, using the fact that ln is continuous (Theorem 2.8),

$$\lim_{n\to\infty} \ln\big(\ln(15n+28)\big) = \ln\Big(\ln\big(\lim_{n\to\infty}(15n+28)\big)\Big) = \infty.$$

so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\ln\big(\ln(15n+28)\big)} = 0.$$

By Theorem 1.48

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln\big(\ln(15n+28)\big)}$$

is convergent.

D.11. $\boxed{\textit{Relevant page(s) in the notes: } 24}$

First note that

$$\frac{1}{n^2-62n+961} = \frac{1}{(n-31)^2}.$$

Therefore, the 31st term in the series has the form

$$\frac{1}{0} \quad \text{which does not correspond to a finite number.}$$

More formally,

$$\frac{1}{(31+\varepsilon-31)^2} = \frac{1}{\varepsilon^2} \to \infty \quad \text{as} \quad \varepsilon \to 0.$$

Since all other the terms in the series

$$\sum_{n=1}^{\infty} \frac{1}{(n-31)^2}$$

are positive and finite, the series cannot converge. Hence, the series in the exercise is divergent.

D.12. $\boxed{\textit{Relevant page(s) in the notes: } 21}$

Let

$$a_n = \frac{g^n}{n!}.$$

Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{g^{n+1}/(n+1)!}{g^n/n!} = \frac{g}{n+1} = \frac{10^{10^{100}}}{n+1}.$$

Therefore,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 10^{10^{100}} \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

so by the Ratio Test for Series (Theorem 1.41) the series

$$\sum_{n=0}^{\infty} \frac{g^n}{n!} = \sum_{n=0}^{\infty} \frac{10^{n10^{100}}}{n!}$$

converges.

D.13. Relevant page(s) in the notes: 18, 19

This is publication-quality proof. See † at the end of this document for a full explanation.

Let

$$a_n = \frac{(33+31n^3+7n^2)^{1/4}}{(10n^{17}+11n^{25}+3n^3+98)^{1/14}} \quad \text{and} \quad b_n = n^{-29/28}, \quad n \in \mathbb{N}.$$

Then

$$\begin{aligned}
\frac{a_n}{b_n} &= \frac{(33+31n^3+7n^2)^{1/4} n^{29/28}}{(10n^{17}+11n^{25}+3n^3+98)^{1/14}} \\
&= \frac{(33+31n^3+7n^2)^{1/4} n^{-3/4}}{(10n^{17}+11n^{25}+3n^3+98)^{1/14} n^{-25/14}} \\
&= \frac{\{(33+31n^3+7n^2)n^{-3}\}^{1/4}}{\{(10n^{17}+11n^{25}+3n^3+98)n^{-25}\}^{1/14}} \\
&= \frac{(33n^{-3}+31+7n^{-1})^{1/4}}{(10n^{-8}+11+3n^{-22}+98n^{-25})^{1/14}}.
\end{aligned}$$

Hence

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{31^{1/4}}{11^{1/14}} \neq 0.$$

Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} n^{-29/28} \quad \text{converges,}$$

by the Limit Comparison Test (Theorem 1.40)

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(33+31n^3+7n^2)^{1/4}}{(10n^{17}+11n^{25}+3n^3+98)^{1/14}} \quad \text{also converges.}$$

Preliminary background working that led to an effective b_n choice

The numerator of a_n is dominated by $(31n^3)^{1/4}$ which is proportional to $n^{3/4}$.

The denominator of a_n is dominated by $(11n^{25})^{1/14}$ which is proportional to $n^{25/14}$.

Therefore, for large n, a_n is proportional to

$$\frac{n^{3/4}}{n^{25/14}} = n^{-29/28}.$$

This suggests that we do a Limit Comparison Test with $b_n = n^{-29/28}$.

D.14. Relevant page(s) in the notes: 18

First note that

$$\begin{aligned}
n! &= n(n-1)(n-2)\cdots(4)(3)(2)(1) \\
&< n(n-0)(n-0)\cdots(n-0)(n-0)(2)(1) \\
&= 2n^{n-2}.
\end{aligned}$$

Therefore,

$$\frac{n!}{n^n} < \frac{2n^{n-2}}{n^n} = \frac{2}{n^2} \quad \text{for all } n \in \mathbb{N}.$$

Since

$$\sum_{n=1}^{\infty} \frac{2}{n^2} = \frac{\pi^2}{3}$$

and therefore is convergent, by the Comparison Test (Theorem 1.39)

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} \quad \text{is also convergent.}$$

D.15. *Relevant page(s) in the notes:* 18

This is publication-quality proof. See † at the end of this document for a full explanation.

First we check whether or not all terms in

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - 798n + 113} \tag{9}$$

are finite. The quadratic equation

$$x^2 - 798x + 113 = 0 \quad \text{has roots} \quad x = \tfrac{1}{2}(798 \pm \sqrt{\Delta}) \quad \text{where} \quad \Delta = 636352.$$

Since

$$797^2 = 635209 < \Delta < 636804 = 798^2$$

Δ is not a perfect square, which implies that neither of the roots of this quadratic equation are integers. Therefore the series in (9) has all terms finite.

Let $N = 1596$ and then note that

$$n \geq N \quad \Longrightarrow \quad \frac{n}{2} \geq 798 \quad \Longrightarrow \quad \frac{n}{2} - 798 \geq 0 \quad \Longrightarrow \quad n\left(\frac{n}{2} - 798\right) \geq 0. \tag{10}$$

The last inequality in (10) is equivalent to

$$\frac{n^2}{2} - 798n \geq 0 \quad \Longrightarrow \quad \frac{n^2}{2} - 798n + 113 > 0 \quad \Longrightarrow \quad n^2 - 798n + 113 > \frac{n^2}{2}.$$

Therefore,

$$\text{for all} \quad n \geq 1596 \quad \text{we have} \quad \frac{1}{n^2 - 798n + 113} < \frac{2}{n^2}.$$

Since

$$\sum_{n=1}^{\infty} \frac{2}{n^2} = \frac{\pi^2}{3} \quad \text{and, therefore, is convergent}$$

by the Comparison Test (Theorem 1.39)

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - 798n + 113} \quad \text{is also convergent.}$$

Preliminary background working that led to an effective N choice

Since the series with nth term

$$\frac{1}{n^2 - 798n + 113} = \frac{1}{n^2(1 - 798/n + 113/n^2)} \tag{11}$$

it is anticipated that its behaviour is similar to the series with nth term $1/n^2$. However, bounding the sequence in (11) with $2/n^2$ is much easier and the factor of 2 does not impact series convergences. The plan is to use the Comparison Test (Theorem 1.39) which means that we need to find N such that

$$n \geq N \quad \Longrightarrow \quad \frac{1}{n^2 - 798n + 113} \leq \frac{2}{n^2} \quad \Longleftrightarrow \quad \tfrac{1}{2}\, n^2 - 798n + 113 \geq 0.$$

The quadratic equation

$$\tfrac{1}{2}x^2 - 798x + 113 = 0 \quad \textit{has solutions} \quad x = 798 \pm \sqrt{798^2 - 226}$$

Since the the coefficient of x^2 is positive, the corresponding parabola is U-shaped and so

$$n \geq 798 + \sqrt{798^2 - 226} \quad \Longrightarrow \quad \tfrac{1}{2}\, n^2 - 798n + 113 \geq 0.$$

Since $\sqrt{798^2 - 226} < \sqrt{798^2} = 798$ we can also say that

$$n > 798 + 798 \quad \Longrightarrow \quad \tfrac{1}{2}\, n^2 - 798n + 113 \geq 0.$$

Therefore, setting $N = 798 + 798 = 1596$ is sufficient for getting to

$$n \geq N \quad \Longrightarrow \quad \frac{1}{n^2 - 798n + 113} \leq \frac{2}{n^2}.$$

† According to the lecturer's experiences in publishing proofs in mathematics and statistics journals over many years, preliminary background working **should not be included** in the proof. This is the convention in esteemed journals such as *The Annals of Mathematics* and *The Annals of Statistics*. Many web-sites and some books **do not follow** publication-quality conventions and tend to include such preliminary background working. Subject participants are encouraged to learn about and use publication-quality versions of mathematical proofs.

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