UNIVERSITY OF TECHNOLOGY SYDNEY

School of Mathematical and Physical Sciences

# 35007 Real Analysis

## EXERCISE SET C – Solutions

**THEME: Sequence Limits**

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C.1. *Relevant page(s) in the notes:* 12

(a)

$$\lim_{n\to\infty} \frac{7n^3+2n^2-n+16}{4n^3+98n^2+31n-11} = \lim_{n\to\infty} \frac{7+2/n-1/n^2+16/n^3}{4+98/n+31/n^2-11/n^3} = \frac{7}{4}.$$

(b) Let

$$a_n \equiv \frac{-1}{\sqrt[3]{19n+42}} \quad \text{and} \quad c_n \equiv \frac{1}{\sqrt[3]{19n+42}} \quad \text{and note that} \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = 0.$$

Since

$$a_n \le \frac{(-1)^n}{\sqrt[3]{19n+42}} \le c_n$$

by the Sandwich Theorem (Theorem 1.23)

$$\lim_{n\to\infty} \frac{(-1)^n}{\sqrt[3]{19n+42}} = 0.$$

(c)

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{\sqrt{n}(\sqrt{n+1}-\sqrt{n})} &= \lim_{n\to\infty} \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n}(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})} \\
&= \lim_{n\to\infty} \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n}(n+1-n)} \\
&= \lim_{n\to\infty} \left(\sqrt{1+1/n}+1\right) \\
&= 2.
\end{aligned}$$

C.2. *Relevant page(s) in the notes:* Supplementary Results

Let

$$a_n \equiv \frac{974^n}{(n+33)!}.$$

[1]This document is less than a few months old which makes errors more likely. Updated versions with later dates may be released during the semester. Notifications will be sent via e-mail whenever an updated version is released.

Then all $a_n \neq 0$ and

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{974^{n+1}}{(n+34)!} \times \frac{(n+33)!}{974^n} = \frac{974}{n+34} = \frac{974/n}{1+34/n}.$$

Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{974/n}{1+34/n}\right) = 0.$$

So by the Ratio Test for Sequences (Theorem S.1 of the Supplementary Results)

$$\lim_{n\to\infty}\frac{974^n}{(n+33)!} = 0.$$

C.3. $\boxed{\textit{Relevant page(s) in the notes: } 12}$

Define the sequences

$$a_n \equiv -\frac{1}{(n+4)^{1/39}}, \quad \text{and} \quad c_n \equiv \frac{1}{(n+4)^{1/39}} \quad n \in \mathbb{N}.$$

Then for all $n \in \mathbb{N}$,

$$-1 \leq \cos(n) \leq 1 \quad \Longrightarrow \quad -1 \leq \cos^{703}(n) \leq 1 \quad \Longrightarrow \quad a_n \leq \frac{\cos^{703}(n)}{(n+4)^{1/39}} \leq c_n. \quad (1)$$

Now

$$\lim_{n\to\infty} a_n = -\left\{\lim_{n\to\infty}\left(\frac{1}{n+4}\right)\right\}^{1/39} = 0^{1/39} = 0$$

and

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty}(-a_n) = -\lim_{n\to\infty}(a_n) = 0.$$

Because of (1) and the Sandwich Theorem (Theorem 1.23)

$$\lim_{n\to\infty}\frac{\cos^{703}(n)}{(n+4)^{1/39}} = 0.$$

C.4. $\boxed{\textit{Relevant page(s) in the notes: } 12}$

(a) From the result concerning the function $h$ with $x = \sqrt{n}$, for all $n \in \mathbb{N}$,

$$\sqrt{n} - 2\ln(\sqrt{n}) > 0 \quad \Longrightarrow \quad \sqrt{n} - \ln(n) > 0 \quad \Longrightarrow \quad \sqrt{n} > \ln(n).$$

(b) Define the sequences

$$a_n = 0 \quad \text{and} \quad c_n = \frac{1}{\sqrt{n}}, \quad n \in \mathbb{N}.$$

Firstly,

$$n \geq 1 \quad \Longrightarrow \quad \ln(n) \geq 0 \quad \Longrightarrow \quad \frac{\ln(n)}{n} \geq 0 = a_n. \quad (2)$$

Secondly, from (a),

$$\ln(n) < \sqrt{n} \quad \Longrightarrow \quad \frac{\ln(n)}{n} < \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} = c_n. \tag{3}$$

Combining (2) and (3) then gives

$$a_n \leq \frac{\ln(n)}{n} < c_n \quad \text{for all } n \in \mathbb{N}.$$

Also,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 0 = 0 \quad \text{and} \quad \lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

So, by the Sandwich Theorem (Theorem 1.23),

$$\lim_{n\to\infty} \frac{\ln(n)}{n} = 0.$$

C.5. *Relevant page(s) in the notes:* 10

With division of the numerator and denominator by $n^3$ we have

$$\begin{aligned}\lim_{n\to\infty} \frac{(3n+8419)(7n+2768)(2n+3452)}{18n^3+93275} &= \lim_{n\to\infty} \frac{(3+8419/n)(7+2768/n)(2+3452/n)}{18+93275/n^3} \\ &= \frac{(3)(7)(2)}{18} \\ &= \frac{7}{3}.\end{aligned}$$

C.6. *Relevant page(s) in the notes:* 21, 22, 39, 40, Supplementary Results

The following cases are treated separately:

(i) $x = 1$, (ii) $x = -1$, (iii) $|x| < 1$ and (iv) $|x| > 1$.

For each $n \in \mathbb{N}$

$$f_n(1) = \frac{1+1^n}{1+1^n} = 1 \quad \text{so, trivially,} \quad \lim_{n\to\infty} f_n(1) = 1.$$

Thus, $f_n(x)$ converges for $x = 1$.

For each $n \in \mathbb{N}$

$$f_n(-1) = \frac{-1+(-1)^n}{1+(-1)^n} = \begin{cases} -\infty & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

Therefore

$$\liminf_{n\to\infty} f_n(-1) = -\infty \quad \text{and} \quad \limsup_{n\to\infty} f_n(-1) = 0.$$

Since

$$-\infty = \liminf_{n\to\infty} f_n(-1) \neq \limsup_{n\to\infty} f_n(-1) = 0$$

$f_n(-1)$ diverges (Proposition 1.44). Thus, $f_n(x)$ does not converge for $x = -1$.

If $|x| < 1$ then $\lim_{n\to\infty} x^n = 0$ and so

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x + x^n}{1 + x^n} = x.$$

Thus, $f_n(x)$ converges for $|x| < 1$.

If $|x| > 1$ then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x + x^n}{1 + x^n} = \lim_{n\to\infty} \frac{x^{1-n} + 1}{x^{-n} + 1} = \frac{1 + \lim_{n\to\infty} (x^{1-n})}{1 + \lim_{n\to\infty} (x^{-n})}.$$

Application Ratio Test for Sequences leads to

$$\lim_{n\to\infty} (x^{1-n}) = 0 \quad \text{and} \quad \lim_{n\to\infty} (x^{-n}) = 0 \quad \text{for all} \quad |x| > 1.$$

Therefore,

$$\lim_{n\to\infty} f_n(x) = 1 \quad \text{for all} \quad |x| > 1.$$

In summary $f_n(x)$ converges if and only if $x \neq -1$.

C.7. $\boxed{\textit{Relevant page(s) in the notes: } 10}$

Noting the identity

$$a^2 - b^2 = (a-b)(a+b) \quad \text{for all } a, b \in \mathbb{R}$$

we have, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\sqrt{2n^2 + 7n + 3} - \sqrt{2n^2 - 4n + 9} \\
&\quad = \frac{\left(\sqrt{2n^2 + 7n + 3} - \sqrt{2n^2 - 4n + 9}\right)\left(\sqrt{2n^2 + 7n + 3} + \sqrt{2n^2 - 4n + 9}\right)}{\sqrt{2n^2 + 7n + 3} + \sqrt{2n^2 - 4n + 9}} \\
&\quad = \frac{(2n^2 + 7n + 3) - (2n^2 - 4n + 9)}{\sqrt{2n^2 + 7n + 3} + \sqrt{2n^2 - 4n + 9}} \\
&\quad = \frac{11n - 6}{\sqrt{2n^2 + 7n + 3} + \sqrt{2n^2 - 4n + 9}} \\
&\quad = \frac{11 - 6/n}{\sqrt{2 + 7/n + 3/n^2} + \sqrt{2 - 4/n + 9/n^2}}.
\end{aligned}$$

Hence

$$\begin{aligned}
\lim_{n\to\infty} \left(\sqrt{2n^2 + 7n + 3} - \sqrt{2n^2 - 4n + 9}\right) &= \lim_{n\to\infty} \left(\frac{11 - 6/n}{\sqrt{2 + 7/n + 3/n^2} + \sqrt{2 - 4/n + 9/n^2}}\right) \\
&= \frac{11}{2\sqrt{2}} \\
&= \frac{11\sqrt{2}}{4}.
\end{aligned}$$

C.8. $\boxed{\textit{Relevant page(s) in the notes: } 10}$

Noting the identity

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2) \quad \text{for all } a, b \in \mathbb{R}$$

we have, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&(7n^3 + 93n^2 - 11n + 8)^{1/3} - (7n^3 + 57n^2 - 54n + 29)^{1/3} \\
&= \big\{(7n^3 + 93n^2 - 11n + 8)^{1/3} - (7n^3 + 57n^2 - 54n + 29)^{1/3}\big\} \\
&\quad \times \big\{(7n^3 + 93n^2 - 11n + 8)^{2/3} + (7n^3 + 93n^2 - 11n + 8)^{1/3}(7n^3 + 57n^2 - 54n + 29)^{1/3} \\
&\qquad + (7n^3 + 57n^2 - 54n + 29)^{2/3}\big\} \\
&\quad \times \big\{(7n^3 + 93n^2 - 11n + 8)^{2/3} + (7n^3 + 93n^2 - 11n + 8)^{1/3}(7n^3 + 57n^2 - 54n + 29)^{1/3} \\
&\qquad + (7n^3 + 57n^2 - 54n + 29)^{2/3}\big\}^{-1} \\
&= \big\{(7n^3 + 93n^2 - 11n + 8) - (7n^3 + 57n^2 - 54n + 29)\big\} \\
&\quad \times \big\{(7n^3 + 93n^2 - 11n + 8)^{2/3} + (7n^3 + 93n^2 - 11n + 8)^{1/3}(7n^3 + 57n^2 - 54n + 29)^{1/3} \\
&\qquad + (7n^3 + 57n^2 - 54n + 29)^{2/3}\big\}^{-1} \\
&= \big(36n^2 + 43n - 21\big) \\
&\quad \times \big\{(7n^3 + 93n^2 - 11n + 8)^{2/3} + (7n^3 + 93n^2 - 11n + 8)^{1/3}(7n^3 + 57n^2 - 54n + 29)^{1/3} \\
&\qquad + (7n^3 + 57n^2 - 54n + 29)^{2/3}\big\}^{-1} \\
&= \big(36 + 43/n - 21/n^2\big) \\
&\quad \times \big\{(7 + 93/n - 11/n^2 + 8/n^3)^{2/3} \\
&\qquad\quad + (7 + 93/n - 11/n^2 + 8/n^3)^{1/3}(7 + 57/n - 54/n^2 + 29/n^3)^{1/3} \\
&\qquad\quad + (7 + 57/n - 54/n^2 + 29/n^3)^{2/3}\big\}^{-1}.
\end{aligned}$$

From this last expression it follows that

$$\lim_{n\to\infty} \big\{(7n^3 + 93n^2 - 11n + 8)^{1/3} - (7n^3 + 57n^2 - 54n + 29)^{1/3}\big\} = \frac{12}{7^{2/3}} = \frac{12\sqrt[3]{7}}{7}.$$

C.9. $\boxed{\textit{Relevant page(s) in the notes: } 10}$

Using the algebraic identity $a^2 - b^2 = (a+b)(a-b)$ we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sqrt[3]{n+2}\,\sqrt[3]{n^2+2n+4}}{(n+1)^2-(n-1)^2} &= \lim_{n\to\infty} \frac{\sqrt[3]{n+2}\,\sqrt[3]{n^2+2n+4}}{(2n)(2)} \\
&= \lim_{n\to\infty} \frac{\sqrt[3]{(n+2)(n^2+2n+4)}}{4n} \\
&= \lim_{n\to\infty} \sqrt[3]{\frac{(n+2)(n^2+2n+4)}{4^3\,n^3}} \\
&= \lim_{n\to\infty} \sqrt[3]{\frac{(1+2/n)(1+2/n+4/n^2)}{4^3}} \\
&= \lim_{n\to\infty} \sqrt[3]{\frac{1}{4^3}} \\
&= \frac{1}{4}.
\end{aligned}$$

C.10. $\boxed{\textit{Relevant page(s) in the notes:}\text{ Supplementary Results}}$

Let

$$a_n \equiv \frac{(n+9)!17^{12n+n^2}}{(2n+3)!} \quad \text{for } n \in \mathbb{N}.$$

Then

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{(n+10)!17^{12(n+1)+(n+1)^2}}{(2n+5)!} \times \frac{(2n+3)!}{(n+9)!17^{12n+n^2}} \\
&= \frac{(n+10)17^{12n+12+n^2+2n+1-12n-n^2}}{(2n+5)(2n+4)} \\
&= \frac{(n+10)17^{2n+13}}{(2n+5)(2n+4)} \\
&= \frac{17^{13}(1+10/n)}{(2+5/n)(2+4/n)}\,\frac{\exp\{2n\ln(17)\}}{n} \\
&= \frac{17^{13}(1+10/n)}{(2+5/n)(2+4/n)}\,b_n
\end{aligned}$$

where

$$b_n \equiv \frac{\exp\{2n\ln(17)\}}{n}.$$

Since

$$\left|\frac{b_{n+1}}{b_n}\right| = \frac{\exp\{2(n+1)\ln(17)\}}{n+1} \times \frac{n}{\exp\{2n\ln(17)\}} = \frac{17^2}{1+1/n} = \frac{289}{1+1/n}$$

we have

$$\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = 289 > 1$$

so by the Ratio Test for Sequences (Theorem S.1 of the Supplementary Results) $b_n$ is divergent. Hence

$$\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$$

and, by another application of the Ratio Test for Sequences (Theorem S.1), $a_n$ is divergent.

C.11. $\boxed{\textit{Relevant page(s) in the notes:}\ \text{Supplementary Results}}$

Let

$$a_n \equiv \frac{(n+9)!17^{12n}}{(2n+3)!} \quad \text{for } n \in \mathbb{N}.$$

Then

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{(n+10)!17^{12(n+1)}}{(2n+5)!} \times \frac{(2n+3)!}{(n+9)!17^{12n}} \\
&= \frac{(n+10)17^{12}}{(2n+5)(2n+4)} \\
&= \frac{(1+10/n)}{(2+5/n)(2+4/n)} \times \frac{17^{12}}{n}.
\end{aligned}$$

Therefore,

$$L \equiv \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(1+10/n)}{(2+5/n)(2+4/n)} \times \lim_{n\to\infty}\frac{17^{12}}{n} = 0 < 1.$$

Since $L < 1$, by the Ratio Test for Sequences (Theorem S.1 of the Supplementary Results)

$$\lim_{n\to\infty}\frac{(n+9)!17^{12n}}{(2n+3)!} = 0.$$

C.12. $\boxed{\textit{Relevant page(s) in the notes:}\ \text{Supplementary Results}}$

With division of the numerator and denominator by $3^n$ we have

$$\lim_{n\to\infty}\frac{(-2)^n+3^n}{(-2)^{n+1}+3^{n+1}} = \lim_{n\to\infty}\frac{(-2/3)^n+1}{(-2)(-2/3)^n+3}$$

Using the Ratio Test for Sequences (Theorem S.1 of the Supplementary Results)

$$\lim_{n\to\infty}(-2/3)^n = 0$$

we then obtain

$$\lim_{n\to\infty}\frac{(-2)^n+3^n}{(-2)^{n+1}+3^{n+1}} = \frac{1}{3}.$$

C.13. $\boxed{\textit{Relevant page(s) in the notes: } 10, 26}$

$$\lim_{n\to\infty}\left\{\left(8-\frac{1}{\sqrt[n]{2}}\right)\frac{3n^3+5}{2n^3+n^2-1}\right\}=\left\{8-\lim_{n\to\infty}\exp\left(\frac{-\ln(2)}{n}\right)\right\}\lim_{n\to\infty}\frac{3+5/n^3}{2+1/n-1/n^3}$$
$$=\{8-\exp(0)\}(3/2)$$
$$=\frac{21}{2}.$$

C.14. $\boxed{\textit{Relevant page(s) in the notes: } 12}$

First note that
$$\tanh(x)=\frac{\sinh(x)}{\cosh(x)}=\frac{e^x-e^{-x}}{e^x+e^{-x}}.$$
For $x\geq 0$ we then have
$$\tanh(x)=\frac{e^{2x}-1}{e^{2x}+1}.$$
Since, for $x\geq 0$,
$$0\leq e^{2x}-1<e^{2x}+1 \quad\Longrightarrow\quad 0\leq\frac{e^{2x}-1}{e^{2x}+1}<1 \iff 0\leq\tanh(x)<1 \quad \text{for all} \quad x\geq 0.$$
For $x<0$ we have
$$\tanh(x)=\frac{1-e^{-2x}}{1+e^{-2x}} \qquad\Longrightarrow\qquad -\tanh(x)=\frac{e^{-2x}-1}{e^{-2x}+1}.$$
Since, for $x<0$,
$$0<e^{-2x}-1<e^{-2x}+1 \implies 0<\frac{e^{-2x}-1}{e^{-2x}+1}<1 \iff 0<-\tanh(x)<1$$
$$\iff -1<\tanh(x)<0 \quad \text{for all} \quad x<0.$$
Combining these findings for both the $x\geq 0$ and $x<0$ cases we have
$$-1<\tanh(x)<1 \quad \text{for all } x\in\mathbb{R}. \tag{4}$$

Define the sequences
$$a_n=\frac{13n^2+7n+84}{19n^2+14n+10}, \quad \text{and} \quad c_n=\frac{13n^2+7n+86}{19n^2+14n+10} \quad n\in\mathbb{N}.$$
Then, using (4),
$$-1\leq\tanh\left((95n!)^{87014^{n!}}\right)\leq 1 \quad\Longrightarrow\quad -1\leq\tanh^{n^2}\left((95n!)^{87014^{n!}}\right)\leq 1$$
$$\Longrightarrow\; 13n^2-1+7n+85\leq 13n^2+\tanh^{n^2}\left((95n!)^{87014^{n!}}\right)+7n+85\leq 13n^2+1+7n+85$$
$$\Longrightarrow\; \frac{13n^2-1+7n+85}{19n^2+14n+10}\leq\frac{13n^2+\tanh^{n^2}\left((95n!)^{87014^{n!}}\right)+7n+85}{19n^2+14n+10}\leq\frac{13n^2+1+7n+85}{19n^2+14n+10}$$
$$\iff a_n\leq\frac{13n^2+\tanh^{n^2}\left((95n!)^{87014^{n!}}\right)+7n+85}{19n^2+14n+10}\leq c_n.$$

Next note that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{13n^2+7n+84}{19n^2+14n+10} = \lim_{n\to\infty} \frac{13+7/n+84/n^2}{19+14/n+10/n^2} = \frac{13}{19}$$

and

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{13n^2+7n+86}{19n^2+14n+10} = \lim_{n\to\infty} \frac{13+7/n+86/n^2}{19+14/n+10/n^2} = \frac{13}{19}.$$

So, by Sandwich Theorem (Theorem 1.23),

$$\lim_{n\to\infty} \frac{13n^2+\tanh^{n^2}\left((95n!)^{87014^{n!}}\right)+7n+85}{19n^2+14n+10} = \frac{13}{19}.$$

C.15. $\boxed{\textit{Relevant page(s) in the notes: } 10, 26}$

We have

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{3984+211^n} &= \lim_{n\to\infty} (3984+211^n)^{1/n} \\
&= \lim_{n\to\infty} \exp\left[\ln\{(3984+211^n)^{1/n}\}\right] \\
&= \lim_{n\to\infty} \exp\left[\{\ln(3984+211^n)\}/n\right] \\
&= \exp\left\{\lim_{n\to\infty} \frac{\ln(3984+211^n)}{n}\right\} \\
&= \exp\left\{\lim_{n\to\infty} \frac{\ln\left(211^n\left(\frac{3984}{211^n}+1\right)\right)}{n}\right\} \\
&= \exp\left\{\lim_{n\to\infty} \frac{n\ln(211)+\ln\left(\frac{3984}{211^n}+1\right)}{n}\right\} \\
&= \exp\left[\ln(211)+\ln\left(\lim_{n\to\infty}\left(\frac{3984}{211^n}+1\right)\right)\lim_{n\to\infty}\frac{1}{n}\right] \\
&= \exp\left\{\ln(211)+\ln(1)(0)\right\} \\
&= 211.
\end{aligned}$$

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