UNIVERSITY OF TECHNOLOGY SYDNEY

School of Mathematical and Physical Sciences

35007 Real Analysis

EXERCISE SET B – Solutions

THEME: **Sequences Theory**

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The Floor and Ceiling Notation. These solutions use a less standard notation which is common in mathematical journals and some mathematics textbooks. For $x \in \mathbb{R}$ the floor notation is

$$\lfloor x \rfloor = \text{greatest integer less than or equal to } x = \text{“floor” of } x$$

and the ceiling notation is

$$\lceil x \rceil = \text{smallest integer greater than or equal to } x = \text{“ceiling” of } x.$$

Examples of this notation are

$$\lfloor 5.78 \rfloor = 5, \quad \lceil 5.78 \rceil = 6, \quad \lfloor 34 \rfloor = 34, \quad \lceil 34 \rceil = 34, \quad \lfloor -8.46 \rfloor = -9, \quad \lceil -8.46 \rceil = -8.$$

B.1. *Relevant page(s) in the notes:* 8

We treat the following two cases separately:

(i) $c = 0$ and (ii) $c \neq 0$.

First suppose that $c = 0$. Then for all $\varepsilon > 0$

$$n \geq 1 \quad \Longrightarrow \quad |ca_n - ca| = |0 - 0| = 0 < \varepsilon.$$

Therefore, by Definition 1.16,

$$\lim_{n\to\infty} (ca_n) = ca \quad \text{when} \quad c = 0.$$

Now suppose that $c \neq 0$. Since

$$\lim_{n\to\infty} a_n = a \tag{1}$$

for any $\varepsilon > 0$ it is also true that $\varepsilon/|c| > 0$ and so, by (1), there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \Longrightarrow \quad |a_n - a| < \varepsilon/|c|.$$

[1] This document is less than a few months old which makes errors more likely. Updated versions with later dates may be released during the semester. Notifications will be sent via e-mail whenever an updated version is released.

Hence,

$$n \geq N \quad \Longrightarrow \quad |ca_n - ca| = |c||a_n - n| < |c|(\varepsilon/|c|) = \varepsilon$$

and so by Definition 1.16

$$\lim_{n\to\infty} (ca_n) = ca \quad \text{when} \quad c \neq 0.$$

B.2. *Relevant page(s) in the notes: 8*

This is publication-quality proof. See † at the end of this document for a full explanation.

For any given $\varepsilon > 0$ set $N = \lceil 7/\varepsilon \rceil$. Then $n \geq N$ implies that

$$n \geq 7/\varepsilon \implies n\varepsilon \geq 7 \implies n\varepsilon + 2\varepsilon > 7 \implies 7/(n+2) < \varepsilon.$$

Noting that $7/(n+2) = |3(n-1)/(n+2) - 3|$ for all $n \in \mathbb{N}$ we then have

$$n \geq N \implies \left| \frac{3n-1}{n+2} - 3 \right| < \varepsilon.$$

Therefore, by Definition 1.16,

$$\lim_{n\to\infty} \frac{3n-1}{n+2} = 3.$$

Preliminary background working that led to an effective N choice

For each $\varepsilon > 0$ need to get to

$$\left| \frac{3n-1}{n+2} - 3 \right| < \varepsilon. \tag{2}$$

Simplification of left-hand side of (2) leads to the equivalent condition

$$\frac{7}{n+2} < \varepsilon \quad \Longleftrightarrow \quad n > 7/\varepsilon - 2$$

The simpler condition $n \geq 7/\varepsilon$ implies $n > 7/\varepsilon - 2$ and avoids having to deal with the possibility of negative numbers on the right-hand side (e.g. if $\varepsilon = 10$). So setting

$$N = \lceil 7/\varepsilon \rceil = \textit{the smallest integer greater than or equal to } 7/\varepsilon$$

will allow us to say that $n \geq N$ implies (2).

B.3. *Relevant page(s) in the notes: 6, 8*

This is publication-quality proof. See † at the end of this document for a full explanation.

For any given $\varepsilon > 0$ set

$$N = \left\lceil \sqrt{24/(49\varepsilon)} \right\rceil.$$

Then $n \geq N$ implies that, with use of the triangle inequality,

$$n \geq \sqrt{24/(49\varepsilon)} \implies 49\varepsilon n^2 \geq 24 \implies 49\varepsilon n^3 \geq 24n \geq 21 + 3n \geq |21 - 3n|.$$

However, for all $n \in \mathbb{N}$,

$$49\varepsilon n^3 \geq |21-3n| \quad \Longrightarrow \quad 49\varepsilon n^3 + 21n\varepsilon > |21-3n| \quad \Longleftrightarrow \quad \left|\frac{21-3n}{49\varepsilon n^3 + 21n}\right| < \varepsilon.$$

Noting that

$$\frac{21-3n}{49n^3+21n} = \frac{n^3+3}{7n^3+3n} - \frac{1}{7} \quad \text{for all } n \in \mathbb{N}$$

we then have

$$n \geq N \implies \left|\frac{n^3+3}{7n^3+3n} - \frac{1}{7}\right| < \varepsilon.$$

Therefore, by Definition 1.16,

$$\lim_{n\to\infty} \frac{n^3+3}{7n^3+3n} = \frac{1}{7}.$$

Preliminary background working that led to an effective N choice

For each $\varepsilon > 0$ need to get to

$$\left|\frac{n^3+3}{7n^3+3n} - \frac{1}{7}\right| < \varepsilon \tag{3}$$

Some simple algebraic steps lead to

$$\frac{n^3+3}{7n^3+3n} - \frac{1}{7} = \frac{|21-3n|}{49n^3+21n}.$$

By the triangle inequality, for all $n \in \mathbb{N}$,

$$|21-3n| \leq 21 + 3n \leq 21n + 3n = 24n$$

Also, for all $n \in \mathbb{N}$,

$$49n^3 + 21n > 49n^3 \quad \Longrightarrow \quad \frac{1}{49n^3+21n} < \frac{1}{49n^3}.$$

On combining these last three equations

$$\left|\frac{n^3+3}{7n^3+3n} - \frac{1}{7}\right| < \frac{24n}{49n^3} = \frac{24}{49n^2}.$$

Therefore having n big enough so that $24/(49n^2) \leq \varepsilon$ implies (3) for such n. Therefore, the choice $N = \left\lceil \sqrt{24/(49\varepsilon)} \right\rceil$ is such that $n \geq N$ leads to what is needed to arrive at (3).

B.4. *Relevant page(s) in the notes:* 8

We treat the following three cases separately:

(i) $a = 0$, (ii) $a > 0$ and (iii) $a < 0$.

First suppose that $a = 0$. Since

$$\lim_{n\to\infty} a_n = 0 \tag{4}$$

for any $\varepsilon > 0$ it is also true that $\varepsilon^2 > 0$ and so, by (4), there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \Longrightarrow \quad |a_n| < \varepsilon^2 \quad \Longrightarrow \quad \sqrt{|a_n|} < \varepsilon \quad \Longleftrightarrow \quad \left|\sqrt{a_n} - \sqrt{0}\right| < \varepsilon. \tag{5}$$

Therefore, by Definition 1.16

$$\lim_{n\to\infty} \sqrt{a_n} = \sqrt{a} \quad \text{when} \quad a = 0.$$

Now suppose that $a > 0$. Since

$$\lim_{n\to\infty} a_n = a \tag{6}$$

for any $\varepsilon > 0$ it is also true that $\varepsilon\sqrt{a} > 0$ and so, by (6), there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \Longrightarrow \quad |a_n - a| < \varepsilon\sqrt{a}. \tag{7}$$

Next note that

$$a_n - a = (\sqrt{a_n})^2 - (\sqrt{a})^2 = (\sqrt{a_n} - \sqrt{a})(\sqrt{a_n} + \sqrt{a}) \quad \Longrightarrow \quad \sqrt{a_n} - \sqrt{a} = \frac{a_n - a}{\sqrt{a_n} + \sqrt{a}}. \tag{8}$$

Hence, using (7) and (8),

$$n \geq N \quad \Longrightarrow \quad \left|\sqrt{a_n} - \sqrt{a}\right| = \frac{|a_n - a|}{\sqrt{a_n} + \sqrt{a}} \leq \frac{|a_n - a|}{\sqrt{a}} < \frac{\varepsilon\sqrt{a}}{\sqrt{a}} = \varepsilon.$$

Therefore, by Definition 1.16

$$\lim_{n\to\infty} \sqrt{a_n} = \sqrt{a} \quad \text{when} \quad a > 0.$$

Finally, we show that $a < 0$ is not possible when (6) holds with all $a_n \geq 0$. We do this by contradiction. If $a < 0$ then $\varepsilon = -a/2 > 0$. Then, from (6) there is $N \in \mathbb{N}$ such that

$$n \geq N \quad \Longrightarrow \quad |a_n - a| < -a/2 \quad \Longleftrightarrow \quad a/2 < a_n - a < -a/2 \quad \Longrightarrow a_n < a/2 < 0.$$

But this $a_n < 0$ result contradicts the $a_n \geq 0$ assumption. Therefore, $a < 0$ is not possible and this case can be excluded.

B.5. *Relevant page(s) in the notes:* 7, 8

We will use proof by contradiction. Suppose that

$$\lim_{n\to\infty} n = L \quad \text{for some } L \in \mathbb{R}.$$

Then from the ε-N definition of a limit (Definition 1.16) with $\varepsilon = 1$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \Longrightarrow \quad |n - L| < 1 \quad \Longleftrightarrow \quad L - 1 < n < L + 1 \quad \Longrightarrow \quad n < L + 1. \tag{9}$$

It follows from (9) for all $n \geq N$ we have $n < L + 1$. However, this would imply that $\lfloor L + 1 \rfloor$ is the largest natural number, which is a contradiction of the Archimedean Property. Therefore the sequence $a_n = n,\ n \in \mathbb{N}$, diverges.

B.6. *Relevant page(s) in the notes:* 8

This is publication-quality proof. See † at the end of this document for a full explanation.

For each $\varepsilon > 0$ set $N = \lceil 361/(7\varepsilon^2) \rceil$. Then $n \geq N$ implies that

$$n \geq 361/(7\varepsilon^2) \quad \implies \quad \frac{361}{7n} \leq \varepsilon^2 \quad \iff \quad \frac{19}{\sqrt{7n}} \leq \varepsilon. \tag{10}$$

Next note that

$$7n + 29 > 7n \quad \implies \quad \sqrt{7n+29} > \sqrt{7n} \quad \implies \quad \sqrt{7n+29} + \sqrt{7n+10} > \sqrt{7n}$$

which then implies that

$$\frac{1}{\sqrt{7n+29} + \sqrt{7n+10}} < \frac{1}{\sqrt{7n}}. \tag{11}$$

Combining (10) and (11) we obtain

$$\frac{19}{\sqrt{7n+29} + \sqrt{7n+10}} < \varepsilon \iff \frac{19\big(\sqrt{7n+29} - \sqrt{7n+10}\big)}{\big(\sqrt{7n+29} + \sqrt{7n+10}\big)\big(\sqrt{7n+29} - \sqrt{7n+10}\big)} < \varepsilon. \tag{12}$$

Application of the $(a+b)(a-b) = a^2 - b^2$ identity to the denominator and some simple algebra leads to the second inequality in (12) being equivalent to

$$\Big|\big(\sqrt{7n+29} - \sqrt{7n+10}\big) - 0\Big| < \varepsilon.$$

Therefore, from Definition 1.16,

$$\lim_{n\to\infty} \big(\sqrt{7n+29} - \sqrt{7n+10}\big) = 0.$$

<u>*Preliminary background working that led to an effective N choice*</u>

For each $\varepsilon > 0$ need to get to

$$\Big|\sqrt{7n+29} - \sqrt{7n+10} - 0\Big| < \varepsilon \quad \iff \quad \sqrt{7n+29} - \sqrt{7n+10} < \varepsilon. \tag{13}$$

Multiplication of both sides of the second inequality in (13) by $\sqrt{7n+29} + \sqrt{7n+10}$, use of $a^2 - b^2 = (a-b)(a+b)$ and simplification then gets us to

$$19 < \varepsilon(\sqrt{7n+29} + \sqrt{7n+10}) \quad \iff \quad \frac{19}{\sqrt{7n+29} + \sqrt{7n+10}} < \varepsilon. \tag{14}$$

Since (as shown above)

$$\frac{19}{\sqrt{7n+29} + \sqrt{7n+10}} < \frac{19}{\sqrt{7n}}$$

the simpler condition

$$\frac{19}{\sqrt{7n}} < \varepsilon \tag{15}$$

implies the second condition in (14). Simple algebra shows that condition (15) is equivalent to

$$n > \frac{361}{7\varepsilon^2}$$

which means that taking $N = \lceil 361/(7\varepsilon^2) \rceil$ will be sufficient.

B.7. $\boxed{\textit{Relevant page(s) in the notes: } 7, 8}$

We will use proof by contradiction. Suppose that

$$\lim_{n\to\infty} (-1)^n n = L \quad \text{for some } L \in \mathbb{R}.$$

Then from the ε-N definition of a limit (Definition 1.16) with $\varepsilon = 1$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \Longrightarrow \quad \left|(-1)^n n - L\right| < 1 \quad \Longleftrightarrow \quad L - 1 < (-1)^n n < L + 1. \tag{16}$$

The last condition in (16) implies that

$$\begin{aligned} n &< 1 + L \quad \text{when } n \text{ is even and} \\ n &< 1 - L \quad \text{when } n \text{ is odd.} \end{aligned}$$

Therefore, for all $n \geq N$ we have $n < \max(1 - L, 1 + L)$. However, this would imply that $\lfloor \max(1 - L, 1 + L) \rfloor$ is the largest natural number, which is a contradiction of the Archimedean Property. Therefore the sequence $a_n = (-1)^n n,\ n \in \mathbb{N}$, diverges.

B.8. $\boxed{\textit{Relevant page(s) in the notes: } 13}$

Since

$$|a_n| = |\sin(n)|\,|\cos(n^2 + 1)| \leq 1 \quad \text{for all } n \in \mathbb{N}$$

a_n is a bounded sequence. Therefore, from the Bolzano-Weierstrass Theorem (Theorem 1.26), a_n has a convergent subsequence.

B.9. $\boxed{\textit{Relevant page(s) in the notes: } 6, 13}$

Let

$$a_n \equiv \frac{\{1 + (-1)^n\}(5n^2 + 7)}{6n^2 + n + 3} + \frac{\{1 + (-1)^{n+1}\}(19n - 11)}{13n^2 + 8}, \qquad n \in \mathbb{N}.$$

Then, for all $n \in \mathbb{N}$, from the triangle inequality

$$\begin{aligned} |a_n| &\leq \frac{|1 + (-1)^n|(5n^2 + 7)}{6n^2 + n + 3} + \frac{|1 + (-1)^{n+1}||19n - 11|}{13n^2 + 8} \\ &\leq \frac{(1 + |(-1)^n|)(5n^2 + 7)}{6n^2 + n + 3} + \frac{(1 + |(-1)^{n+1}|)(19n + 11)}{13n^2 + 8} \\ &= \frac{2(5n^2 + 7)}{6n^2 + n + 3} + \frac{2(19n + 11)}{13n^2 + 8}. \end{aligned}$$

Noting that for all $n \in \mathbb{N}$

$$5n^2 + 7 \leq 5n^2 + 7n^2 = 12n^2 \quad \text{and} \quad 6n^2 + n + 3 > 6n^2 \quad \Longrightarrow \quad \frac{1}{6n^2 + n + 3} < \frac{1}{6n^2}$$

we have

$$\frac{2(5n^2 + 7)}{6n^2 + n + 3} < \frac{12n^2}{6n^2} = 2 \quad \text{for all } n \in \mathbb{N}.$$

Similar arguments lead to

$$\frac{19n + 11}{13n^2 + 8} < \frac{30}{13} < 3 \quad \text{for all } n \in \mathbb{N}.$$

Therefore,
$$|a_n| < 2 \times 2 + 2 \times 3 = 10 \quad \text{for all } n \in \mathbb{N}$$
and so the sequence in the exercise is bounded. Next note that
$$a_n = \begin{cases} \dfrac{2(19/n^2 - 11/n)}{13 + 8/n^2} & n \text{ odd,} \\[2ex] \dfrac{2(5 + 7/n^2)}{6 + 1/n + 3/n^2} & n \text{ even.} \end{cases}$$
The subsequence of a_n corresponding to n odd is a convergent subsequence since it converges to 0. Alternatively, the subsequence of a_n corresponding to n even is a convergent subsequence since it converges to $5/3$.

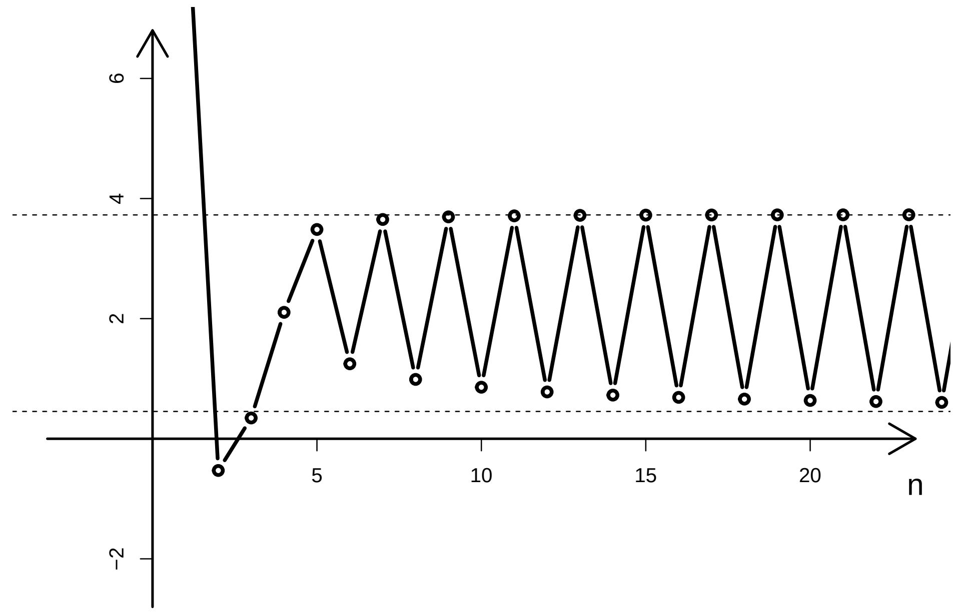
B.10. *Relevant page(s) in the notes:* 22

First note that
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{19 + 15/n + 17/n^2}{41 - 22/n - 109/n^2} \right) = \frac{19}{41}.$$
which means that a_n is convergent. By Proposition 1.44 we must have
$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \frac{19}{41}.$$

B.11. *Relevant page(s) in the notes:* 21

Some simple algebraic steps lead to
$$a_n = 2(-1)^{n+1} + \frac{(-1)^n(4 + 7/n - 3/n^2)}{11 - 29/n} + \frac{23 + 5/n}{11 - 7/n} \quad \text{for all } n \in \mathbb{N}.$$
Then note that
$$a_n = \begin{cases} 2 - \dfrac{4 + 7/n - 3/n^2}{11 - 29/n} + \dfrac{23 + 5/n}{11 - 7/n}, & n \text{ odd,} \\[2ex] -2 + \dfrac{4 + 7/n - 3/n^2}{11 - 29/n} + \dfrac{23 + 5/n}{11 - 7/n}, & n \text{ even.} \end{cases}$$
The subsequence of a_n corresponding to n odd converges to
$$2 - \frac{4}{11} + \frac{23}{11} = \frac{41}{11}.$$
The subsequence of a_n corresponding to n even converges to
$$-2 + \frac{4}{11} + \frac{23}{11} = \frac{5}{11}.$$
However, as is apparent from the accompanying figure, all convergent subsequences fall into one of two cases:

- those that are, ultimately, subsequences of the $\{a_n : n \text{ odd}\}$ subsequence which converge to $41/11$,
- those that are, ultimately, subsequences of the $\{a_n : n \text{ even}\}$ subsequence which converge to $5/11$.



Therefore

$$\liminf_{n\to\infty} a_n = \frac{5}{11} \quad \text{and} \quad \limsup_{n\to\infty} a_n = \frac{41}{11}.$$

B.12. *Relevant page(s) in the notes:* 21

Some simple algebraic steps lead to

$$a_n = \sqrt{n}\{1-(-1)^n\} + \frac{\{1+(-1)^n\}(9/n+14/n^2)}{13/n+23} \quad \text{for all } n \in \mathbb{N}.$$

Then note that

$$a_n = \begin{cases} 2\sqrt{n}, & n \text{ odd}, \\ \dfrac{2(9/n+14/n^2)}{13/n+23}, & n \text{ even}. \end{cases}$$

The subsequence of a_n corresponding to n odd is unbounded and diverges to ∞. Therefore

$$\limsup_{n\to\infty} a_n = \text{the largest limit among } a_n\text{'s subsequences} = \infty.$$

The subsequence of a_n corresponding to n even converges to 0. Moreover, the only convergent subsequence are those that are, ultimately, subsequences of the $\{a_n : n \text{ even}\}$ subsequence which converge to 0. Therefore,

$$\liminf_{n\to\infty} a_n = 0.$$

In summary,

$$\liminf_{n\to\infty} a_n = 0 \quad \text{and} \quad \limsup_{n\to\infty} a_n = \infty.$$

B.13. $\boxed{\textit{Relevant page(s) in the notes: } 13}$

For each $\varepsilon > 0$ let $N = \lceil 1/\varepsilon \rceil$. Then let n and m be natural numbers such that $n, m \geq N$. Then

$$n, m \geq 1/\varepsilon \quad \implies \frac{1}{n}, \frac{1}{m} \leq \varepsilon.$$

In the case where $n = m$ we have

$$\left| \frac{1}{n} - \frac{1}{m} \right| = 0 < \varepsilon.$$

In the case where $n > m$ we have

$$\frac{1}{n} < \frac{1}{m} \quad \implies \quad \frac{1}{m} - \frac{1}{n} > 0 \quad \implies \quad \left| \frac{1}{m} - \frac{1}{n} \right| = \frac{1}{m} - \frac{1}{n} < \frac{1}{m} \leq \varepsilon.$$

In the case where $m > n$ analogous arguments lead to

$$\left| \frac{1}{m} - \frac{1}{n} \right| < \varepsilon.$$

So regardless of whether $m = n$, $n > m$ or $n < m$

$$n, m \geq N \quad \implies |a_n - a_m| < \varepsilon$$

so by Definition 1.27 a_n is a Cauchy sequence.

B.14. $\boxed{\textit{Relevant page(s) in the notes: } 6, 13}$

This is publication-quality proof. See † at the end of this document for a full explanation.

For each $\varepsilon > 0$ set $N = \lceil (6/\varepsilon)^{3/2} \rceil$. Then $n \geq N$ implies that

$$n \geq (6/\varepsilon)^{3/2} \quad \implies \quad 3n^{-2/3} \leq \varepsilon/2 \tag{17}$$

Next note that

$$\begin{aligned} n + 2 > n \quad &\implies \quad (n+2)^{2/3} > n^{2/3} \\ &\implies \quad (n+5)^{2/3} + (n+5)^{1/3}(n+2)^{1/3} + (n+2)^{2/3} > n^{2/3} \end{aligned}$$

which then implies that

$$\frac{1}{(n+5)^{2/3} + (n+5)^{1/3}(n+2)^{1/3} + (n+2)^{2/3}} < n^{-2/3}. \tag{18}$$

Combining (17) and (18)

$$\begin{aligned} &\frac{3}{(n+5)^{2/3} + (n+5)^{1/3}(n+2)^{1/3} + (n+2)^{2/3}} < \varepsilon/2 \\ \iff &\frac{3\{(n+5)^{1/3} - (n+2)^{1/3}\}}{\{(n+5)^{2/3} + (n+5)^{1/3}(n+2)^{1/3} + (n+2)^{2/3}\}\{(n+5)^{1/3} - (n+2)^{1/3}\}} < \varepsilon/2. \end{aligned} \tag{19}$$

Application of the identity $(a^2+ab+b^2)(a-b)=a^3-b^3$ to the denominator and some simple algebra leads to the second inequality in (19) being equivalent to

$$(n+5)^{1/3}-(n+2)^{1/3}<\varepsilon/2 \quad\iff\quad |a_n|<\varepsilon/2.$$

Thus, we have shown that

$$n\geq N \quad \text{implies that} \quad |a_n|<\varepsilon/2. \tag{20}$$

Similarly,

$$m\geq N \quad \text{implies that} \quad |a_m|<\varepsilon/2. \tag{21}$$

From (20) and (21) and using the triangle inequality

$$n,m\geq N \qquad\Longrightarrow\qquad |a_n-a_m|\leq|a_n|+|a_m|<\varepsilon/2+\varepsilon/2=\varepsilon.$$

Therefore, from Definition 1.27, a_n is a Cauchy sequence.

Preliminary background working that led to an effective N choice

We need to get to

$$n,m\geq N \qquad\Longrightarrow\qquad |a_n-a_m|<\varepsilon \tag{22}$$

where $a_n=\sqrt[3]{n+5}-\sqrt[3]{n+2}$. Intuitively, the two terms of a_n get closer as $n\to\infty$ so it is likely that a_n converges to zero. This suggest applying the triangle inequality as follows:

$$|a_n-a_m|\leq|a_n|+|a_m|=|a_n-0|+|a_m-0| \tag{23}$$

and finding N such that, for any $\varepsilon>0$

$$n\geq N \qquad\Longrightarrow\qquad |a_n|<\varepsilon/2$$

By symmetry, a similar result will hold for $m\geq N$ and substitution into (23) will then give us (22).

So now we need to get to

$$|a_n|<\varepsilon/2 \quad\iff\quad 2\{(n+5)^{1/3}-(n+2)^{1/3}\}<\varepsilon. \tag{24}$$

Multiplication of both sides of the second inequality in (24) by

$$(n+5)^{2/3}+(n+5)^{1/3}(n+2)^{1/3}+(n+2)^{2/3},$$

use of $(a^2+ab+b^2)(a-b)=a^3-b^3$ and simplification leads to

$$\begin{aligned} &6<\varepsilon\{(n+5)^{2/3}+(n+5)^{1/3}(n+2)^{1/3}+(n+2)^{2/3}\}\\ \iff\qquad &\frac{6}{(n+5)^{2/3}+(n+5)^{1/3}(n+2)^{1/3}+(n+2)^{2/3}}<\varepsilon. \end{aligned} \tag{25}$$

Since (based on (18) above),

$$\frac{6}{(n+5)^{2/3}+(n+5)^{1/3}(n+2)^{1/3}+(n+2)^{2/3}}<6n^{-2/3}.$$

the simpler condition

$$6n^{-2/3}<\varepsilon \tag{26}$$

implies the second condition in (25). Since (26) is equivalent to $n>(6/\varepsilon)^{3/2}$, setting

$$N=\lceil(6/\varepsilon)^{3/2}\rceil$$

will suffice.

B.15. $\boxed{\textit{Relevant page(s) in the notes: } 13}$

We will use proof by contradiction. Suppose that $a_n = n$ is a Cauchy sequence. Then from the N-ε definition of a Cauchy sequence (Definition 1.27) with $\varepsilon = \frac{1}{2}$, there exists $N \in \mathbb{N}$ such that

$$m, n \geq N \quad \Longrightarrow \quad |m - n| < \tfrac{1}{2} \quad \Longleftrightarrow \quad m = n.$$

However, $m = N$ and $n = N + 1$ satisfy $m, n \geq N$ but are such that $m \neq n$, which is a contradiction. Hence, no such N exists and so a_n is not a Cauchy sequence.

† According to the lecturer's experiences in publishing proofs in mathematics and statistics journals over many years, preliminary background working **should not be included** in the proof. This is the convention in esteemed journals such as *The Annals of Mathematics* and *The Annals of Statistics*. Many web-sites and some books **do not follow** publication-quality conventions and tend to include such preliminary background working. Subject participants are encouraged to learn about and use publication-quality versions of mathematical proofs.

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