UNIVERSITY OF TECHNOLOGY SYDNEY

School of Mathematical and Physical Sciences

35007 Real Analysis

EXERCISE SET A – Solutions

THEME: Real Analysis Basic Concepts

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A.1. (a) Since $c > 0$ we have

$$0 < a < b \quad \Longrightarrow \quad 0 < ac < bc. \tag{1}$$

Also, since $b > 0$ we have

$$0 < c < d \quad \Longrightarrow \quad 0 < bc < bd. \tag{2}$$

On combining the second conditions in each of (1) and (2) we obtain

$$0 < ac < bc < bd \quad \Longrightarrow \quad ac < bd.$$

(b) Let $a = -1$, $b = 1$, $c = -2$ and $d = 1$. Then

$$-1 = a < b = 1 \quad \text{and} \quad -2 = c < d = 1.$$

However,

$$2 = ac > bd = 1.$$

Therefore,

$$a < b \quad \text{and} \quad c < d \quad \text{does not imply} \quad ac < bd.$$

A.2. (a) Since $a, b > 0$ we have $ab > 0$ and then $1/(ab) > 0$. Since the direction of an inequality is not changed when both sides are multiplied by a positive number we have

$$0 < a < b \quad \Longrightarrow \quad \frac{a}{ab} < \frac{b}{ab} \quad \Longrightarrow \quad \frac{1}{b} < \frac{1}{a}.$$

(b) Consider $a = -1$ and $b = 1$. We have $a < b$ but

$$1 = \frac{1}{b} > \frac{1}{a} = -1.$$

A.3. *Relevant page(s) in the notes:* 4

Let

$$X = \{\ldots, -7, -5, -3, -1, 1, 3, 5, 7, \ldots\}$$

[1]This document is less than a few months old which makes errors more likely. Updated versions with later dates may be released during the semester. Notifications will be sent via e-mail whenever an updated version is released.

be the set of odd integers and define the function f on X by

$$f(x) = \begin{cases} x, & x > 0, \\ 1 - x, & x < 0, \end{cases} \qquad x \in X.$$

Then,

$$f(1) = 1, \quad f(3) = 3, \quad f(5) = 5, \quad f(7) = 7, \ldots$$

and

$$f(-1) = 2, \quad f(-3) = 4, \quad f(-5) = 6, \quad f(-7) = 8, \ldots$$

which shows that $f : X \to \mathbb{N}$. Also, for any $x_1, x_2 \in X$,

$$f(x_1) = f(x_2) \quad \Longrightarrow \quad x_1 = x_2$$

which means that f is a one to one function. By Definition 1.6, X is countable.

A.4. $\boxed{\textit{Relevant page(s) in the notes: } 40}$

(a) f maps the character strings in X, which correspond to animal types, to character strings in Y corresponding to the word usually used to describe young animals of that type. From Definition 3.17 f^{-1} must satisfy

$$f^{-1}(f(\texttt{bear})) = \texttt{bear} \quad \Longleftrightarrow \quad f^{-1}(\texttt{cub}) = \texttt{bear}.$$

Similarly $f^{-1}(\texttt{kid}) = \texttt{goat}$ and $f^{-1}(\texttt{cygnet}) = \texttt{swan}$.
In summary, $f^{-1} : Y \to X$ is given by

$$f^{-1}(\texttt{cub}) = \texttt{bear}, \quad f^{-1}(\texttt{kid}) = \texttt{goat}, \quad f^{-1}(\texttt{cygnet}) = \texttt{swan}.$$

(b) From Definition 3.17, f^{-1} must satisfy, for all $x \in \mathbb{R}$,

$$f^{-1}(f(x)) = x \quad \Longleftrightarrow \quad f^{-1}\big(e^x/(1+e^x)\big) = x. \tag{3}$$

Now note that

$$y = e^x/(1+e^x) \iff y = 1/(1+e^{-x}) \iff 1/y = 1+e^{-x} \iff e^x = 1/\{(1/y)-1\}$$

which, after a little more algebra, is equivalent to

$$x = \ln\left(\frac{y}{1-y}\right). \tag{4}$$

Substitution of $y = e^x/(1+e^x)$ and (4) into the last equation of (3) then gives

$$f^{-1}(y) = \ln\left(\frac{y}{1-y}\right) \quad \text{for all} \quad y \in (0,1).$$

(c) From Definition 3.17, f^{-1} must satisfy, for all $x \in \mathbb{R}$,

$$f^{-1}(f(x)) = x \quad \Longleftrightarrow \quad f^{-1}\big(\sqrt[5]{9-11x}\big) = x. \tag{5}$$

Now note that

$$y = \sqrt[5]{9-11x} \quad \Longleftrightarrow \quad y^5 = 9 - 11x \quad \Longleftrightarrow \quad 11x = 9 - y^5$$

which is equivalent to

$$x = \frac{9 - y^5}{11}. \tag{6}$$

Substitution of $y = \sqrt[5]{9 - 11x}$ and (6) into the last equation of (5) then gives

$$f^{-1}(y) = \frac{9 - y^5}{11} \quad \text{for all} \quad y \in \mathbb{R}.$$

A.5. $\boxed{\textit{Relevant page(s) in the notes: } 6}$

The set of interest is $A = \{19, -14, 33, 0, 9, -11\}$ which has a finite number of elements. Clearly

$$-14 = \text{lowest element of } A$$

is a lower bound on A. Also, all other lower bounds on A are less than -14. Therefore

$$-14 = \text{the greatest lower bound on } A = \inf A.$$

Also,

$$33 = \text{highest element of } A$$

is an upper bound on A. Also, all other upper bounds on A are greater than 33. Therefore

$$33 = \text{the least upper bound on } A = \sup A.$$

In summary,

$$\inf A = -14 \quad \text{and} \quad \sup A = 33.$$

A.6. $\boxed{\textit{Relevant page(s) in the notes: } 6}$

As is well-known, the five lowest prime numbers are

$$2, 3, 5, 7 \text{ and } 11.$$

All other prime numbers are higher, so 2 is a lower bound on $\mathbb{P}$. All other lower bounds on $\mathbb{P}$ are less than 2 and so

$$2 = \text{the greatest lower bound on } \mathbb{P} = \inf \mathbb{P}.$$

It is well-known from number theory that there is an infinite number of prime numbers. Therefore, for any real number $x \geq 2$ there exists $p \in \mathbb{P}$, such that $p \geq x$. This implies that no finite real number can be an upper bound on $\mathbb{P}$ and so

$$\infty = \text{the least upper bound on } \mathbb{P} = \sup \mathbb{P}.$$

In summary

$$\inf \mathbb{P} = 2 \quad \text{and} \quad \sup \mathbb{P} = \infty.$$

A.7. $\boxed{\textit{Relevant page(s) in the notes: } 6}$

The set

$$\{m/n : m, n \in \mathbb{N}, m + n \leq 10\}.$$

has a finite number of elements consisting of all unique fractions of integers between 1 and 10 inclusive subject to the constraint that the sum of the numerator and denominator must not exceed 10. Some of the elements are

$$7/4, \quad 2/9, \quad 5/1 \quad \text{and} \quad 5/8.$$

If enumerated and equalities such $1/2 = 2/4 = 3/6 = 4/8 = 5/10$ are taken into account then the set has 31 unique entries. Based on these considerations, it is apparent that the lowest possible fraction in the set is $1/9$ and the highest one is $9/1 = 9$. Therefore

$$\inf\{m/n : m, n \in \mathbb{N}, m + n \leq 10\} = 1/9 \quad \text{and} \quad \sup\{m/n : m, n \in \mathbb{N}, m + n \leq 10\} = 9.$$

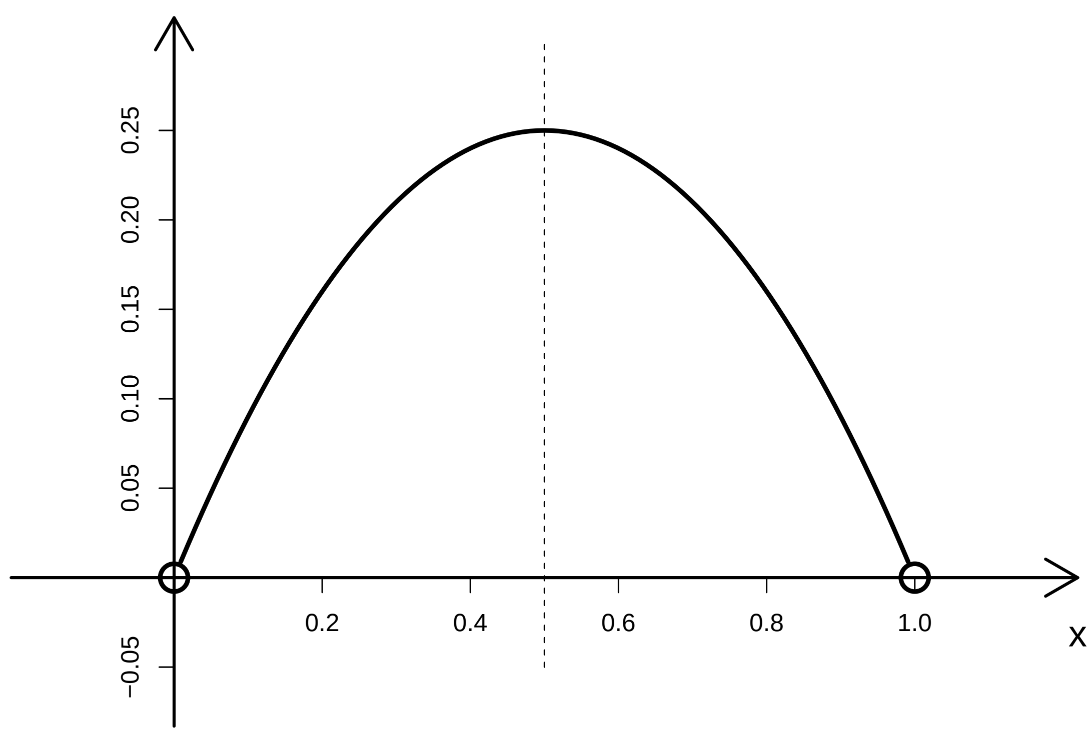
A.8. $\boxed{\textit{Relevant page(s) in the notes: } 6}$

The theory of quadratic functions has established that a function of the form ax^2+bx+c such that $a < 0$ is an arch-shaped (as opposed to bowl-shaped) parabola with axis of symmetry at $x = -b/(2a)$. Therefore the function $x - x^2$ is an arch-shaped parabola with axis of symmetry at $x = \frac{1}{2}$. This implies that the parabola's vertex, and maximum, is at the vertical position

$$\tfrac{1}{2} - (\tfrac{1}{2})^2 = \tfrac{1}{4}.$$

The accompanying figure provides illustration. Since this maximum occurs inside the interval $(0, 1)$ $\frac{1}{4}$ is the maximum value of the set in the exercise. Hence $\frac{1}{4}$ is an upper bound for $\{x - x^2 : 0 < x < 1\}$ and the lowest number that has this property. Thus

$$\sup\{x - x^2 : 0 < x < 1\} = \tfrac{1}{4}.$$



Regarding the infimum, it is straightforward to show and apparent from the accompanying figure that 0 is a lower bound for $\{x - x^2 : 0 < x < 1\}$ and is greater than all negative real numbers (which are also lower bounds). Any $0 < \varepsilon \leq \frac{1}{4}$ is not a lower bound for $\{x - x^2 : 0 < x < 1\}$ since

$$(\varepsilon/2) - (\varepsilon/2)^2 \in \{x - x^2 : 0 < x < 1\} \quad \text{but} \quad \varepsilon > (\varepsilon/2) > (\varepsilon/2) - (\varepsilon/2)^2.$$

Hence

$$\inf\{x - x^2 : 0 < x < 1\} = 0.$$

A.9. *Relevant page(s) in the notes: 6*

This is publication-quality proof. See † at the end of this document for a full explanation.

Let $A = \{n/(n+1) : n \in \mathbb{N}\}$ be the set of interest. Noting that

$$A = \{1/2, 2/3, 3/4, \ldots\}$$

we see that

$$\frac{1}{2} \leq n/(n+1) < 1 \quad \text{for all } n \in \mathbb{N}. \tag{7}$$

Therefore $1/2$ is lower bound on A and greater than any other lower bound (e.g. $1/4$, -7). Hence

$$\inf\{n/(n+1) : n \in \mathbb{N}\} = 1/2.$$

From (7) it is clear that 1 is an upper bound on the set in the exercise. For any $0 < \varepsilon < 1$ let $n_\varepsilon \in \mathbb{N}$ be such that

$$n_\varepsilon > \varepsilon^{-1} - 1.$$

Then

$$1/n_\varepsilon < 1/(\varepsilon^{-1} - 1) \qquad \Longrightarrow \qquad 1 + 1/n_\varepsilon < 1 + 1/(\varepsilon^{-1} - 1) = 1/(1 - \varepsilon).$$

Taking reciprocals on both sides of the last inequality we get

$$1/(1 + 1/n_\varepsilon) > 1 - \varepsilon \qquad \Longleftrightarrow \qquad n_\varepsilon/(n_\varepsilon + 1) > 1 - \varepsilon.$$

Therefore, $1 - \varepsilon$ is not an upper bound for A for any $0 < \varepsilon < 1$. This means that 1 is the lowest, or least, upper bound for A. Hence

$$\sup\{n/(n+1) : n \in \mathbb{N}\} = 1.$$

Preliminary background working on how an appropriate n_ε was obtained

For a given $0 < \varepsilon < 1$, we need find n big enough so that

$$\frac{n}{n+1} > 1 - \varepsilon.$$

Taking reciprocals of both sides gives

$$1 + \frac{1}{n} < \frac{1}{1 - \varepsilon} \qquad \Longleftrightarrow \qquad \frac{1}{n} < \frac{1}{1 - \varepsilon} - 1 = \frac{\varepsilon}{1 - \varepsilon}$$

Taking reciprocals again gives

$$n > \frac{1 - \varepsilon}{\varepsilon} = \varepsilon^{-1} - 1.$$

So taking $n_\varepsilon \in \mathbb{N}$ to be bigger than $\varepsilon^{-1} - 1$ will get us what we need.

A.10. $\boxed{\textit{Relevant page(s) in the notes: } 8, 24}$

Let a_n be the sequence in X given by

$$a_n = 11\tfrac{1}{2} \quad \text{for all } n \in \mathbb{N}.$$

Then, trivially,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 11\tfrac{1}{2} = 11\tfrac{1}{2}.$$

So from Definition 2.1(1), $11\frac{1}{2}$ is a limit point of X.

Let b_n be the sequence in X given by

$$b_n = 14 - 1/n \quad \text{for all } n \in \mathbb{N}.$$

Then each $b_n \in X$ even though $14 \notin X$. However,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \big(14 - 1/n\big) = 14.$$

Therefore, according to Definition 2.1(1), 14 is a limit point of X.

To prove that 6 is an isolated point of X note that for any sequence c_n in X,

$$c_n > 8 \quad \Longrightarrow \quad c_n - 6 > 2 \quad \Longrightarrow \quad |c_n - 6| > 2 \quad \text{for all } n \in \mathbb{N}.$$

Therefore, there can be no $N \in \mathbb{N}$ such that $n > N$ implies $|c_n - 6| < 2$. So by Definition 1.16 it is impossible for sequence c_n in X to converge to 6. Hence 6 is an isolated point of X.

A.11. $\boxed{\textit{Relevant page(s) in the notes: } 26}$

To prove that $A = (-24, 77)$ is open let $a \in A$. Then $-24 < a < 77$ which implies that

$$\varepsilon = \min(a + 24, 77 - a) > 0.$$

Then, firstly,

$$|x-a| < \varepsilon \iff -\varepsilon < x-a < \varepsilon \implies x < a+\varepsilon = a+\min(a+24, 77-a) \le a+77-a = 77.$$

That is, $|x - a| < \varepsilon \implies x < 77$. Secondly,

$$|x-a| < \varepsilon \iff -\varepsilon < x-a < \varepsilon \implies x > a-\varepsilon = a-\min(a+24, 77-a) \ge a-(a+24) = -24.$$

That is, $|x - a| < \varepsilon \implies x > -24$.

Combining the two conclusions in the previous paragraph leads to

$$|x - a| < \varepsilon \implies \quad -24 < x < 77.$$

Therefore $\{x : |x - a| < \varepsilon\} \subseteq A$ and by Definition 2.9 A is open.

A.12. $\boxed{\textit{Relevant page(s) in the notes: } 26}$

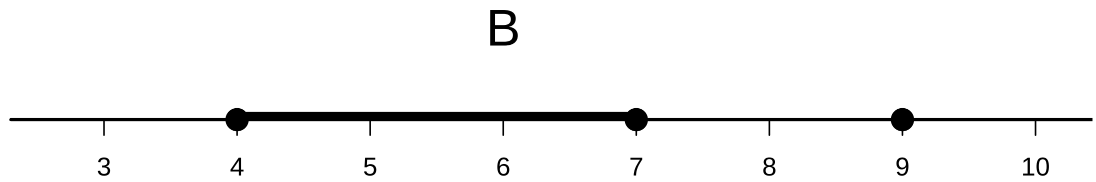
From definitions in and adjacent to Definition 2.9, B is closed if and only if its complement, B^c is open. Note that

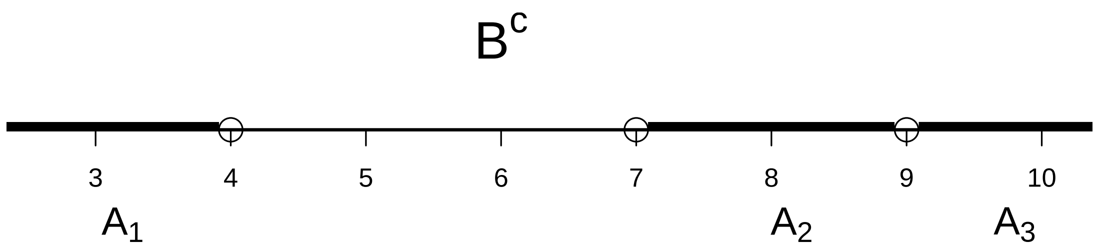
$$B^c = \{x : x < 4\} \cup \{x : 7 < x < 9\} \cup \{x : x > 9\} = A_1 \cup A_2 \cup A_3$$

where

$$A_1 = \{x : x < 4\}, \quad A_2 = \{x : 7 < x < 9\} \quad \text{and} \quad A_3 = \{x : x > 9\}.$$

The accompanying figures provide illustration.





To prove that A_1 is open let $a_1 \in A_1$. Then $a_1 < 4$ which implies that $\varepsilon_1 = 4 - a_1 > 0$. Then

$$|x - a_1| < \varepsilon_1 \iff -\varepsilon_1 < x - a_1 < \varepsilon_1 \implies x < a_1 + \varepsilon_1 = 4 \implies x \in A_1.$$

Therefore $\{x : |x - a_1| < \varepsilon_1\} \subseteq A_1$ and by Definition 2.9 A_1 is open.

To prove that A_2 is open let $a_2 \in A_2$. Then $7 < a_2 < 9$ which implies that

$$\varepsilon_2 = \min(a_2 - 7, 9 - a_2) > 0.$$

Then, firstly,

$$|x-a_2| < \varepsilon_2 \iff -\varepsilon_2 < x-a_2 < \varepsilon_2 \implies x < a_2+\varepsilon_2 = a_2+\min(a_2-7, 9-a_2) \le a_2+9-a_2 = 9.$$

That is, $|x - a_2| < \varepsilon_2 \implies x < 9$. Secondly,

$$|x-a_2| < \varepsilon_2 \iff -\varepsilon_2 < x-a_2 < \varepsilon_2 \implies x > a_2-\varepsilon_2 = a_2-\min(a_2-7, 9-a_2) \ge a_2-(a_2-7) = 7.$$

That is, $|x - a_2| < \varepsilon_2 \implies x > 7$.

Combining the two conclusions in the previous paragraph leads to

$$|x - a_2| < \varepsilon_2 \implies 7 < x < 9.$$

Therefore $\{x : |x - a_2| < \varepsilon_2\} \subseteq A_2$ and by Definition 2.9 A_2 is open.

To prove that A_3 is open let $a_3 \in A_3$. Then $a_3 > 9$ which implies that $\varepsilon_3 = a_3 - 9 > 0$. Then

$$|x - a_3| < \varepsilon_3 \iff -\varepsilon_3 < x - a_3 < \varepsilon_3 \implies x > a_3 - \varepsilon_3 = 9 \implies x \in A_3.$$

Therefore $\{x : |x - a_3| < \varepsilon_3\} \subseteq A_3$ and by Definition 2.9 A_3 is open.

Finally, we need to establish that $A_1 \cup A_2 \cup A_3$ is open. If

$$a \in A_1 \cup A_2 \cup A_3 \quad \text{then either} \quad a \in A_1, \ a \in A_2 \text{ or } a \in A_3.$$

Since A_1 is open, $a \in A_1$ implies that there is $\varepsilon > 0$ such that

$$\{x : |x - a| < \varepsilon\} \subseteq A_1 \subset A_1 \cup A_2 \cup A_3.$$

The conditions $a \in A_2$ and $a \in A_3$ lead to analogous conclusions. Putting these together, $a \in A_1 \cup A_2 \cup A_3$ implies that there is $\varepsilon > 0$ such that

$$\{x : |x - a| < \varepsilon\} \subseteq A_1 \cup A_2 \cup A_3$$

and by Definition 2.9 $A_1 \cup A_2 \cup A_3 = B^c$ is open. Therefore, B is closed.

A.13. $\boxed{\textit{Relevant page(s) in the notes: } 26}$

Let $A \equiv \{\,\}$. From Definition 2.9 A is open since there are no $a \in A$ for which the $B_\varepsilon(a)$ condition needs to hold.

The complement of A is $A^c = \{\,\}^c = \mathbb{R}$. From Definition 2.9 A^c is open since, for any $a \in \mathbb{R}$, the set $B_1(a) \equiv \{x : |x - a| < 1\} = (a - 1, a + 1)$ is contained entirely in $\mathbb{R}$. Since A^c is open, A is closed.

In summary, the empty set $\{\,\}$ is both open and closed.

A.14. $\boxed{\textit{Relevant page(s) in the notes: } 26}$

For each $\varepsilon > 0$ let

$$B_\varepsilon(79) \equiv \{x : |x - 79| < \varepsilon\} = \{x : 79 - \varepsilon < x < 79 + \varepsilon\}.$$

Note that $79 + \frac{1}{2}\varepsilon \in B_\varepsilon(79)$ but $79 + \frac{1}{2}\varepsilon \notin \{79\}$. Therefore, for all $\varepsilon > 0$, $B_\varepsilon(79)$ is not contained in $\{79\}$. Hence $\{79\}$ is not an open set.

The complement of $\{79\}$ is $A_1 \cup A_2$ where

$$A_1 \equiv (-\infty, 79) \quad \text{and} \quad A_2 \equiv (79, \infty).$$

To prove that A_1 is open let $a_1 \in A_1$. Then $a_1 < 79$ which implies that $\varepsilon_1 \equiv 79 - a_1 > 0$. Then

$$|x - a_1| < \varepsilon_1 \iff -\varepsilon_1 < x - a_1 < \varepsilon_1 \implies x < a_1 + \varepsilon_1 = 79 \implies x \in A_1.$$

Therefore $\{x : |x - a_1| < \varepsilon_1\} \subseteq A_1$ and by Definition 2.9 A_1 is open.

To prove that A_2 is open let $a_2 \in A_2$. Then $a_2 > 79$ which implies that $\varepsilon_2 \equiv a_2 - 79 > 0$. Then

$$|x - a_2| < \varepsilon_2 \iff -\varepsilon_2 < x - a_2 < \varepsilon_2 \implies x > a_2 - \varepsilon_2 = 79 \implies x \in A_2.$$

Therefore $\{x : |x - a_2| < \varepsilon_2\} \subseteq A_2$ and by Definition 2.9 A_2 is open.

Finally, we need to establish that $A_1 \cup A_2$ is open. If

$$a \in A_1 \cup A_2 \quad \text{then either} \quad a \in A_1 \text{ or } a \in A_2.$$

Since A_1 is open, $a \in A_1$ implies that there is $\varepsilon > 0$ such that

$$\{x : |x - a| < \varepsilon\} \subseteq A_1 \subset A_1 \cup A_2.$$

The conditions $a \in A_2$ leads to an analogous conclusion. Putting these together, $a \in A_1 \cup A_2$ implies that there is $\varepsilon > 0$ such that

$$\{x : |x - a| < \varepsilon\} \subseteq A_1 \cup A_2$$

and by Definition 2.9 $A_1 \cup A_2 = \{79\}^c$ is open. Therefore, $\{79\}$ is closed.

In summary, $\{79\}$ is closed and not open.

A.15. $\boxed{\textit{Relevant page(s) in the notes: } 26}$

The set of interest is $A = (3,8]$. Clearly $8 \in A$ but, for every $\varepsilon > 0$, the set

$$B_\varepsilon(8) = \{x : |x-8| < \varepsilon\} = \{x : 8 - \varepsilon < x < 8 + \varepsilon\}$$

contains the real number $8 + \frac{1}{2}\varepsilon$ which is not an element of A. Hence, by Definition 2.9, A is not open.

The complement of A is

$$A^c = \{x : x \leq 3\} \cup \{x : x > 8\} = (-\infty, 3] \cup (8, \infty).$$

Clearly $3 \in A^c$. However, for every $0 < \varepsilon < 1$, the set

$$B_\varepsilon(3) = \{x : |x-3| < \varepsilon\} = \{x : 3 - \varepsilon < x < 3 + \varepsilon\}$$

contains the real number $3 + \varepsilon$ which is not an element of A^c. Also, for every $\varepsilon \geq 1$ the set $B_\varepsilon(3)$ contains the number 4, which is not an element of A^c. Hence, for all $\varepsilon > 0$ the set $B_\varepsilon(3)$ is not contained in A^c. So by Definition 2.9, A^c is not open. Since the complement of A is not open, A is not closed.

In summary, $(3,8]$ is neither open nor closed.

† According to the lecturer's experiences in publishing proofs in mathematics and statistics journals over many years, preliminary background working **should not be included** in the proof. This is the convention in esteemed journals such as *The Annals of Mathematics* and *The Annals of Statistics*. Many web-sites and some books **do not follow** publication-quality conventions and tend to include such preliminary background working. Subject participants are encouraged to learn about and use publication-quality versions of mathematical proofs.

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