UNIVERSITY OF TECHNOLOGY SYDNEY

School of Mathematical and Physical Sciences

35007 Real Analysis

EXERCISE SET K

THEME: **Sequences of Integrals**

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Solutions? *Full solutions to all exercises will be released after the relevant class and/or tutorial.*

K.1. Let $M > 0$. Evaluate, with full justification,

$$\lim_{n\to\infty} \int_0^M e^{-n(x^4+1)}\,dx.$$

The fact that each of the integrand functions are continuous and Riemann integrable may be assumed.

K.2. Evaluate, with full justification,

$$\lim_{n\to\infty} \int_{-37}^{88} \frac{(-1)^n \sin^{293}\big(n^7\tanh(x)\big)}{\ln\big(\ln(n)\big)}\,dx$$

The fact that each of the integrand functions are continuous and Riemann integrable may be assumed.

K.3. Evaluate, with full justification,

$$\lim_{n\to\infty} \int_1^7 \frac{n}{3nx^2+5n+1187}\,dx.$$

The fact that each of the integrand functions are continuous and Riemann integrable may be assumed.

K.4. Evaluate, with full justification,

$$\lim_{n\to\infty} \int_{-14}^{8} \frac{(n+9)\cosh(x)+\cos(n!x^{43})}{5n+61}\,dx.$$

The fact that each of the integrand functions are continuous and Riemann integrable may be assumed.

[1] This document is less than a few months old which makes errors more likely. Updated versions with later dates may be released during the semester. Notifications will be sent via e-mail whenever an updated version is released.

K.5. Evaluate, with full justification,

$$\lim_{n\to\infty}\int_2^3 \frac{n}{7nx^4+41}\,dx.$$

The fact that each of the integrand functions are continuous and Riemann integrable may be assumed.

K.6. For each $n\in\mathbb{N}$ let $f:(0,1)\to\mathbb{R}$ be given by

$$f_n(x)=nx^{n-1}.$$

Show that

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\,dx.$$

K.7. Let $f_n:\mathbb{R}\to\mathbb{R}$, $n\in\mathbb{N}$ and $f:\mathbb{R}\to\mathbb{R}$ be given by

$$f_n(x)=nxe^{-nx^2} \quad \text{and} \quad f(x)=0.$$

(a) Prove that f_n converges to f pointwise on $\mathbb{R}$.

(b) Prove that f_n does not converge to f uniformly on $\mathbb{R}$.

(c) Determine whether or not

$$\int_{-\infty}^{\infty}\lim_{n\to\infty} f_n(x)\,dx$$

equals

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\,dx.$$

K.8. Evaluate, with full justification,

$$\lim_{n\to\infty}\int_0^1 \frac{x^2}{1+e^{-nx}}\,dx.$$

The fact that each of the integrand functions are continuous and Riemann integrable may be assumed.

K.9. Evaluate, with full justification,

$$\lim_{n\to\infty}\int_{-3}^{11}\left(1+\frac{(-1)^n}{n}\right)(x-e^{-n})^2\,dx.$$

The fact that each of the integrand functions are continuous and Riemann integrable may be assumed.

K.10. Evaluate, with full justification,

$$\lim_{n\to\infty} \int_{-1}^{2} \frac{e^x}{\left(1+e^x+\frac{1}{4\sqrt{n}}\right)^2} dx.$$

The fact that each of the integrand functions are continuous and Riemann integrable may be assumed.

K.11. Let $s > 1$.

(a) Prove that

$$\frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-nx}\, dx.$$

(b) Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx.$$

The fact that all relevant integrand functions are continuous and Riemann integrable may be assumed. In addition, it may be assumed that extensions of Theorems 5.9 and 5.10 to the interval $[0,\infty)$ hold for the relevant sequences of functions.

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