UNIVERSITY OF TECHNOLOGY SYDNEY

School of Mathematical and Physical Sciences

35007 Real Analysis

EXERCISE SET J

THEME: **Riemann Integration Theory**

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Solutions? *Full solutions to all exercises will be released after the relevant class and/or tutorial.*

J.1. Consider the function $f(x) = x$ for $x \in \mathbb{R}$. Prove that f is Riemann integrable on $[0,1]$ and obtain

$$\int_0^1 f(x)\,dx$$

using the Riemann sums definition of a Riemann integral. The summation result

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

may be assumed.

J.2. Consider the function $f(x) = x^3$ for $x \in \mathbb{R}$. Prove that f is Riemann integrable on $[0,1]$ and obtain

$$\int_0^1 f(x)\,dx$$

using the Riemann sums definition of a Riemann integral. The summation result

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

may be assumed.

J.3. Consider the function

$$f(x) = \frac{1}{\sqrt{4x-1}}, \quad x > \tfrac{1}{4}.$$

Using Riemann's Criterion, prove that f is Riemann integrable on $[3,16]$.

J.4. Consider the function $f(x) = x^2$ for $x \in \mathbb{R}$. Prove that f is Riemann integrable on $[8,35]$ and obtain

$$\int_8^{35} f(x)\,dx$$

[1] This document is less than a few months old which makes errors more likely. Updated versions with later dates may be released during the semester. Notifications will be sent via e-mail whenever an updated version is released.

using the Riemann sums definition of a Riemann integral. The summation results

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

may be assumed.

J.5. Consider the function

$$f(x) = \frac{1}{x}, \quad x \neq 0$$

Using the the partition of the interval $[1, 2]$ with end-points in the set:

$$\mathcal{P} = \{1, 2^{1/n}, 2^{2/n}, \ldots, 2^{(n-1)/n}, 2\}$$

prove that f is Riemann integrable on $[1, 2]$ and obtain

$$\int_1^2 f(x)\, dx$$

using the Riemann sums definition of a Riemann integral.

J.6. Let $f : [0, 1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is an irrational number,} \\ 1 & \text{if } x \text{ is a rational number.} \end{cases}$$

Show that f is not Riemann integrable on $[0, 1]$. The fact that, for any $a, b \in \mathbb{R}$ such that $a < b$, the interval $[a, b)$ contains at least one rational number and one irrational number may be used.

J.7. Verify the following example of Fubini's Theorem:

$$\int_0^2 \int_0^4 (x^2 + 3y)\, dx\, dy = \int_0^2 \int_0^4 (x^2 + 3y)\, dy\, dx.$$

J.8. For define the bivariate function $f : [0, 2] \times [0, 1] \to \mathbb{R}$ as follows:

$$f(x, y) = \frac{xy(x^2 - y^2)}{(x^2 + y^2)^3}, \quad 0 \leq x \leq 2 \quad \text{and} \quad 0 \leq y \leq 1.$$

(a) Obtain

$$\int_0^1 \int_0^2 f(x, y)\, dx\, dy.$$

(b) Obtain

$$\int_0^2 \int_0^1 f(x, y)\, dy\, dx.$$

(c) Discuss whether or not the assumptions of Fubini's Theorem hold for f.

(d) Using a judicious combination of coat hanger wire, chicken coop wire and papier-mâché make a precise 1 metre per unit model of the $f(x, y)$ surface over the rectangle $[0, 2] \times [0, 1]$ with vertical walls at the boundaries to make it watertight. Use a measuring jug and water to determine the volume, in cubic metres, under the surface. Volumes above the $f(x, y)$ plane are positive contributions to the overall volume, whereas those below the $f(x, y)$ plane are negative contributions to the overall volume. Determine whether the volume matches the integral in (a), (b) or neither. What is the explanation for this result?

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