UNIVERSITY OF TECHNOLOGY SYDNEY

School of Mathematical and Physical Sciences

35007 Real Analysis

EXERCISE SET I

THEME: **Series of Functions**

Date of this version: **10th July 2026**[1]

Solutions? *Full solutions to all exercises will be released after the relevant class and/or tutorial.*

I.1. Determine the values of $x \in \mathbb{R}$ for which the following series converges:

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n 8^n}.$$

I.2. Noting that $|\cos(t)| \leq 1$ for all $t \in \mathbb{R}$, and $\sum_{n=1}^{\infty} n^{-3/2}$ converges prove that the series

$$\sum_{n=1}^{\infty} \frac{\cos^5(n^{11}\, x)}{n\sqrt{n+9}} \qquad (x \in \mathbb{R})$$

is uniformly convergent on $\mathbb{R}$.

I.3. Determine the values of $x \in \mathbb{R}$ for which the following series converge:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$,

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$.

I.4. The cosine function is defined to be the following power series:

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad (x \in \mathbb{R}).$$

Prove that this series converges for all $x \in \mathbb{R}$.

I.5. Determine the values of $x \in \mathbb{R}$ for which the following series converges:

$$\sum_{n=0}^{\infty} \frac{(n+13)^{5/2} x^n}{3n+4}.$$

[1]This document is less than a few months old which makes errors more likely. Updated versions with later dates may be released during the semester. Notifications will be sent via e-mail whenever an updated version is released.

I.6. Determine the values of $x \in \mathbb{R}$ for which the following series converges:

$$\sum_{n=1}^{\infty} \{\ln(n)\}^n x^n.$$

The fact that $f(x) = \ln(x), x > 0$ is unbounded and monotone increasing may be assumed.

I.7. Determine the values of $x \in \mathbb{R}$ for which the following series converges:

$$\sum_{n=0}^{\infty} \frac{n^2 x^n}{n!}$$

I.8. Prove that the series

$$\sum_{n=1}^{\infty} \frac{e^x(n+17)}{(1+e^x)(5n^4+12n^2+8)} \qquad (x \in \mathbb{R})$$

is uniformly convergent on $\mathbb{R}$.

I.9. Evaluate the series

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \frac{6}{64} + \ldots$$

I.10. Evaluate

$$\sum_{n=1}^{\infty} \frac{n^2}{17^n}.$$

I.11. Evaluate

$$\sum_{n=1}^{\infty} \frac{n^3}{11^n}.$$

I.12. Determine the Taylor series for $f(x) = \cos(3x)\sin(3x)$ about the point $a = \pi/3$ and show that the series is convergent for all $x \in \mathbb{R}$.

I.13. Determine the Taylor series for $f(x) = \cos^2(x)$ about the point $a = 0$ and show that the series is convergent for all $x \in \mathbb{R}$.

I.14. The hyperbolic sine (sinh) and hyperbolic cosine (cosh) functions are defined according to

$$\sinh(x) = \tfrac{1}{2}(e^x - e^{-x}) \quad \text{and} \quad \cosh(x) = \tfrac{1}{2}(e^x + e^{-x}), \quad x \in \mathbb{R}.$$

Determine the Taylor series for $f(x) = \sinh(x)$ about the point $a = 0$ and show that the series is convergent for all $x \in \mathbb{R}$.

I.15. Obtain the Taylor series for $\ln(x)$ about the point $a = 29$ and then determine the values of $x \in \mathbb{R}$ for which the series converges.

I.16. For all $x \in \mathbb{R}$ and $n \in \mathbb{N}$ define

$$f_n(x) = \frac{x}{n(1 + nx^2)}.$$

(a) Show that

$$\sum_{n=1}^{\infty} f_n(x) \quad \text{converges pointwise on } \mathbb{R}.$$

(b) Show that

$$\sum_{n=1}^{\infty} f_n(x) \quad \text{converges uniformly on } \mathbb{R}.$$

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