

35007

Proofs in Real Analysis

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Some Maths Problems

1. Expand $(x + 3)(4x^2 - 9x - 6)$.
2. Differentiate $\cos^2(e^x + 4 \tan(x))$ with respect to x .
3. Obtain $\int_2^6 \frac{1}{3+x} dx$.
4. Invert $\begin{bmatrix} 8 & 18 \\ -5 & 64 \end{bmatrix}$.

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5. Using the ε - N definition of the limit of a sequence, prove that $\lim_{n \rightarrow \infty} \left(\frac{n}{7n+4} \right) = \frac{1}{7}$.

Formal Proofs

Requirements for a formal (“publication-quality”, direct) mathematical proof

1. Starts with a logical proposition that is self-evidently true.
2. Moves from one logical proposition to the next one – using steps that are self-evidently correct – towards, and eventually concluding with, the required result.
3. Reads like a novel or a news article with correct grammar and punctuation.

In 35007 Real Analysis we will use the abbreviations (not found in novels or newspapers):

$\implies$ for “implies”

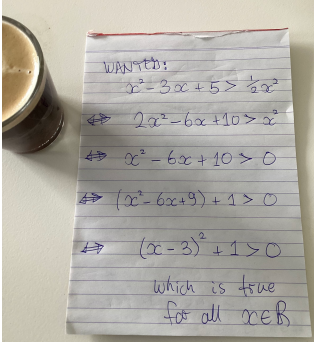
$\iff$ for “if and only if”

Example

Prove that

$$x^2 - 3x + 5 > \frac{1}{2}x^2 \quad \text{for all } x \in \mathbb{R}.$$

The next bit is **NOT** a formal proof (but useful background working):



Wanted:
 $x^2 - 3x + 5 > \frac{1}{2}x^2$
 $\iff 2x^2 - 6x + 10 > x^2$
 $\iff x^2 - 6x + 10 > 0$
 $\iff (x^2 - 6x + 9) + 1 > 0$
 $\iff (x-3)^2 + 1 > 0$
which is true
for all $x \in \mathbb{R}$.

Formal ("Publication-Quality") Proof

The next bit is **NOT** a formal proof (but useful background working):

$$\begin{aligned}
 & \text{Wanted:} \\
 & x^2 - 3x + 5 > \frac{1}{2}x^2 \\
 \Leftrightarrow & 2x^2 - 6x + 10 > x^2 \\
 \Leftrightarrow & x^2 - 6x + 10 > 0 \\
 \Leftrightarrow & (x^2 - 6x + 9) + 1 > 0 \\
 \Leftrightarrow & (x-3)^2 + 1 > 0 \\
 & \text{which is true} \\
 & \text{for all } x \in \mathbb{R}
 \end{aligned}$$

MAIN PROBLEM This does things **BACKWARDS** with respect to our Requirements for a formal ("publication-quality" proof).

For all $x \in \mathbb{R}$

$$(x-3)^2 + 1 > 0$$

Background Working (Without Coffee Cup)

$$\begin{aligned}
 & x^2 - 3x + 5 > \frac{1}{2}x^2 \\
 \Leftrightarrow & 2x^2 - 6x + 10 > x^2 \\
 \Leftrightarrow & x^2 - 6x + 10 > 0 \\
 \Leftrightarrow & x^2 - 6x + 9 + 1 > 0 \\
 \Leftrightarrow & (x-3)^2 + 1 > 0 \\
 & \text{which is true} \\
 & \text{for all } x \in \mathbb{R}
 \end{aligned}$$

Formal ("Publication-Quality") Proof

For all $x \in \mathbb{R}$

$$\begin{aligned}
 & (x-3)^2 + 1 > 0 \\
 \Rightarrow & x^2 - 6x + 10 > 0
 \end{aligned}$$

Formal (“Publication-Quality”) Proof

For all $x \in \mathbb{R}$

$$\begin{aligned} & (x-3)^2 + 1 > 0 \\ \implies & x^2 - 6x + 10 > 0 \\ \implies & 2x^2 - 6x + 10 > x^2 \end{aligned}$$

Formal (“Publication-Quality”) Proof

For all $x \in \mathbb{R}$

$$\begin{aligned} & (x-3)^2 + 1 > 0 \\ \implies & x^2 - 6x + 10 > 0 \\ \implies & 2x^2 - 6x + 10 > x^2 \\ \implies & x^2 - 3x + 5 > \frac{1}{2}x^2. \end{aligned}$$

Therefore,

$$x^2 - 3x + 5 > \frac{1}{2}x^2 \quad \text{for all } x \in \mathbb{R}.$$

Formal (“Publication-Quality”) Proof

For all $x \in \mathbb{R}$

$$\begin{aligned} & (x-3)^2 + 1 > 0 \\ \implies & x^2 - 6x + 10 > 0 \\ \implies & 2x^2 - 6x + 10 > x^2 \\ \implies & x^2 - 3x + 5 > \frac{1}{2}x^2. \end{aligned}$$

IMPORTANT

Publication-quality proofs
separate background working
from the presented proof.

- ***This subject’s exercise solutions*** 
- ***Many web-sites*** 

The Current Solution for Exercise B.2

This is publication-quality proof. See † at the end of this document for a full explanation.

For any given $\varepsilon > 0$ set $N = \lceil 7/\varepsilon \rceil$. Then $n \geq N$ implies that

$$n \geq 7/\varepsilon \implies n\varepsilon \geq 7 \implies n\varepsilon + 2\varepsilon > 7 \implies 7/(n+2) < \varepsilon.$$

Noting that $7/(n+2) = |3(n-1)/(n+2) - 3|$ for all $n \in \mathbb{N}$ we then have

$$n \geq N \implies \left| \frac{3n-1}{n+2} - 3 \right| < \varepsilon.$$

Therefore, by Definition 1.16,

$$\lim_{n \rightarrow \infty} \frac{3n-1}{n+2} = 3.$$

Preliminary background working that led to an effective N choice

For each $\varepsilon > 0$ need to get to

$$\left| \frac{3n-1}{n+2} - 3 \right| < \varepsilon.$$

(1)

Simplification of left-hand side of (1) leads to the equivalent condition

$$\frac{7}{n+2} < \varepsilon \iff n > 7/\varepsilon - 2 \text{ continued...}$$

There is also:

Proof by Contradiction

1. Start by assuming the opposite of the required result.
2. Work towards establishing a falsehood.
3. The final false statement means, by contradiction, that the original assumption is false $\implies$ required result is true.

Roughly 5%–10% of this subject's proofs are by contradiction.

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**INEQUALITIES
AS BUILDING
BLOCKS FOR
REAL ANALYSIS
PROOFS**

QUICK QUIZ

Does

$$a < b \implies a^2 < b^2 \quad ?$$

Does

$$a < b \implies a^2 < b^2 \quad ?$$

ANSWER:

No, since $a = -1$ and $b = 0$ gives an easy counter-example.

$$[-1 < 0 \quad \text{but} \quad (-1)^2 > 0^2]$$

NEXT QUICK QUIZ

Does

$$0 < a < b \implies a^2 < b^2 \quad ?$$

Does

$$0 < a < b \implies a^2 < b^2 \quad ?$$

ANSWER:

Yes!

SECOND LAST QUICK QUIZ

Does

$$a > b \implies \frac{1}{a} < \frac{1}{b} \quad ?$$

SECOND LAST QUICK QUIZ

Does

$$a > b \implies \frac{1}{a} < \frac{1}{b} \quad ?$$

ANSWER:

No, since $a = 1$ and $b = -1$ gives an easy counter-example.

$$[1 > -1 \quad \text{and} \quad 1/1 > 1/(-1)]$$

LAST QUICK QUIZ

Does

$$0 < c < d \implies \frac{1}{d} < \frac{1}{c} \quad ?$$

LAST QUICK QUIZ

Does

$$0 < c < d \implies \frac{1}{d} < \frac{1}{c} \quad ?$$

ANSWER:

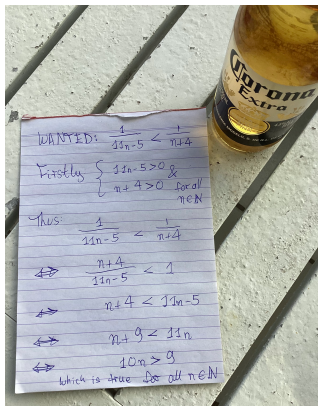
Yes!

Class Exercise

Prove that

$$\frac{1}{11n-5} < \frac{1}{n+4} \quad \text{for all } n \in \mathbb{N}.$$

Class Exercise Background Working (whilst at a pub)



Class Exercise Formal Proof

For all $n \in \mathbb{N}$

$$10n > 9$$

$$\Rightarrow 11n > n + 9$$

$$\Rightarrow 11n - 5 > n + 4 > 0$$

$$\Rightarrow \frac{1}{11n-5} < \frac{1}{n+4}.$$

Therefore,

$$\frac{1}{11n-5} < \frac{1}{n+4} \quad \text{for all } n \in \mathbb{N}.$$

Some Maths Problems Reprise

1. Expand $(x+3)(4x^2-9x-6)$.
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Proving, from First Principles, that a Sequence Converges

Definition 1.16. A sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ is said to be convergent with limit x , if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n - x| < \epsilon$. We write $x_n \rightarrow x$. A sequence which does not converge is said to diverge.

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Closing Example

Using only the ε - N definition (Definition 1.16) **prove** that

$$\frac{1}{n+28} \rightarrow 0 \quad \text{or, equivalently} \quad \lim_{n \rightarrow \infty} \left(\frac{1}{n+28} \right) = 0.$$

Formal Proof (Closing Example)

For each $\varepsilon > 0$ let

$N = \lceil 1/\varepsilon \rceil$ = the smallest integer greater than or equal to $1/\varepsilon$.

Then

$$n \geq N \implies n \geq 1/\varepsilon \implies n + 28 > 1/\varepsilon > 0.$$

Therefore

$$\frac{1}{n+28} < \varepsilon \iff \left| \frac{1}{n+28} - 0 \right| < \varepsilon.$$

So from Definition 1.16,

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n+28} \right) = 0.$$

Getting Ready for the Proof Parts of Quiz 1

- Review these slides.
- Familiarise yourself with Definition 1.16.
- Try Exercises B.1, B.2, B.3 and B.4.
- Attempt Practice Quiz 1.
- Get help from your tutor (Brian or Sam) at the Week 2 tutorial.
- Go through the solutions to be released at 4:00pm on Tuesday 4th August.

