35007

Proofs in Real Analysis

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Some Maths Problems

1. Expand $(x+3)(4x^2-9x-6)$.
2. Differentiate $\cos^2\left(e^x + 4\tan(x)\right)$ with respect to x.
3. Obtain $\displaystyle\int_2^6 \frac{1}{3+x}\,dx$.
4. Invert $\left[\begin{array}{cc} 8 & 18 \\ -5 & 64 \end{array}\right]$.

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5. Using the ε-N definition of the limit of a sequence, prove that $\lim\limits_{n\to\infty}\left(\frac{n}{7n+4}\right)=\frac{1}{7}$.

Formal Proofs

Requirements for a formal ("publication-quality", direct) mathematical proof

1. Starts with a logical proposition that is self-evidently true.
2. Moves from one logical proposition to the next one – using steps that are self-evidently correct – towards, and eventually concluding with, the required result.
3. Reads like a novel or a news article with correct grammar and punctuation.

In 35007 Real Analysis we will use the abbreviations (not found in novels or newspapers):

$$\Longrightarrow \quad \text{for ``implies''}$$

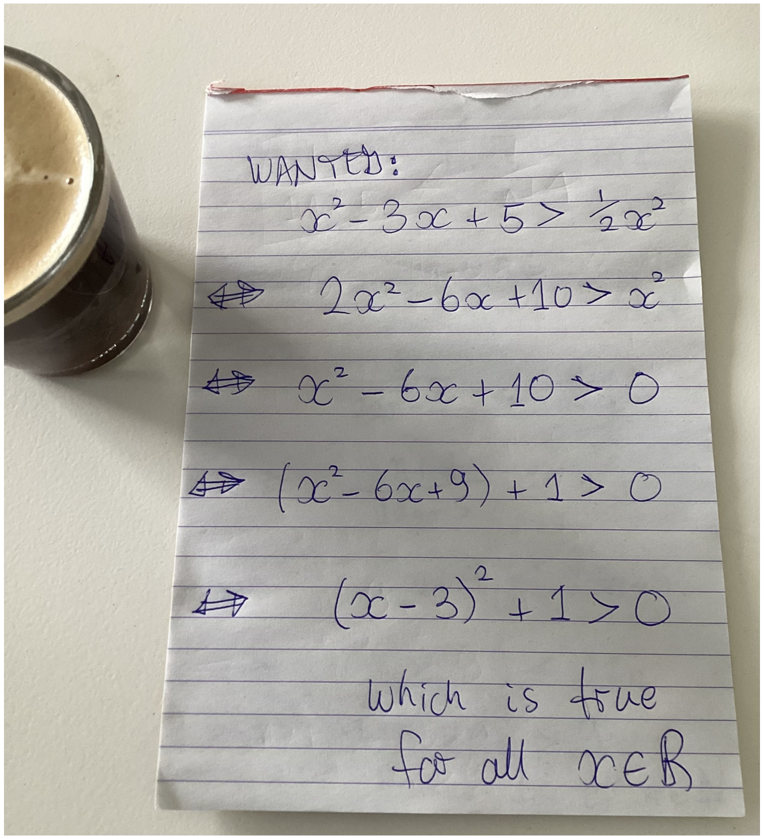
$$\Longleftrightarrow \quad \text{for ``if and only if''}$$

Example

Prove that

$$x^2 - 3x + 5 > \frac{1}{2}x^2 \quad \text{for all } x \in \mathbb{R}.$$

The next bit is NOT a formal proof (but useful background working):



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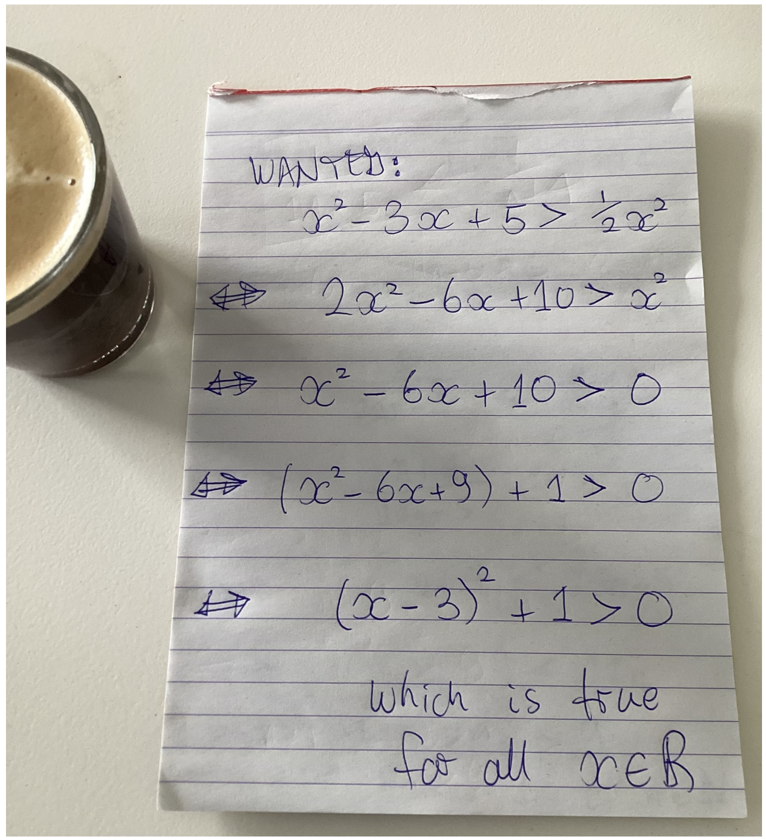

MAIN PROBLEM This does things BACKWARDS with respect to our Requirements for a formal ("publication-quality" proof).

Background Working (Without Coffee Cup)

$$x^2 - 3x + 5 > \tfrac{1}{2}x^2$$

$$\Leftrightarrow \quad 2x^2 - 6x + 10 > x^2$$

$$\Leftrightarrow \quad x^2 - 6x + 10 > 0$$

$$\Leftrightarrow \quad x^2 - 6x + 9 + 1 > 0$$

$$\Leftrightarrow \quad (x-3)^2 + 1 > 0$$

which is true

for all $x \in \mathbb{R}$

Formal ("Publication-Quality") Proof

For all $x \in \mathbb{R}$

$$(x-3)^2 + 1 > 0$$

Formal ("Publication-Quality") Proof

For all $x \in \mathbb{R}$

$$
\begin{aligned}
& (x-3)^2 + 1 > 0 \\
\implies \quad & x^2 - 6x + 10 > 0
\end{aligned}
$$

Formal ("Publication-Quality") Proof

For all $x \in \mathbb{R}$

$$\begin{aligned} & (x-3)^2+1>0 \\ \Longrightarrow \quad & x^2-6x+10>0 \\ \Longrightarrow \quad & 2x^2-6x+10>x^2 \end{aligned}$$

Formal ("Publication-Quality") Proof

For all $x \in \mathbb{R}$

$$
\begin{aligned}
& (x-3)^2 + 1 > 0 \\
\Longrightarrow \quad & x^2 - 6x + 10 > 0 \\
\Longrightarrow \quad & 2x^2 - 6x + 10 > x^2 \\
\Longrightarrow \quad & x^2 - 3x + 5 > \tfrac{1}{2}x^2.
\end{aligned}
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Formal ("Publication-Quality") Proof

For all $x \in \mathbb{R}$

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\Longrightarrow \quad & 2x^2 - 6x + 10 > x^2 \\
\Longrightarrow \quad & x^2 - 3x + 5 > \tfrac{1}{2}x^2.
\end{aligned}
$$

Therefore,

$$x^2 - 3x + 5 > \tfrac{1}{2}x^2 \quad \text{for all } x \in \mathbb{R}.$$

IMPORTANT

Publication-quality proofs

separate background working

from the presented proof.

- ***This subject's exercise solutions*** $\boxed{\checkmark}$
- ***Many web-sites*** $\boxed{\times}$

The Current Solution for Exercise B.2

This is publication-quality proof. See † at the end of this document for a full explanation.

For any given $\varepsilon > 0$ set $N = \lceil 7/\varepsilon \rceil$. Then $n \geq N$ implies that

$$n \geq 7/\varepsilon \implies n\varepsilon \geq 7 \implies n\varepsilon + 2\varepsilon > 7 \implies 7/(n+2) < \varepsilon.$$

Noting that $7/(n+2) = |3(n-1)/(n+2) - 3|$ for all $n \in \mathbb{N}$ we then have

$$n \geq N \implies \left| \frac{3n-1}{n+2} - 3 \right| < \varepsilon.$$

Therefore, by Definition 1.16,

$$\lim_{n\to\infty} \frac{3n-1}{n+2} = 3.$$

Preliminary background working that led to an effective N choice

For each $\varepsilon > 0$ need to get to

$$\left| \frac{3n-1}{n+2} - 3 \right| < \varepsilon. \tag{1}$$

Simplification of left-hand side of (1) leads to the equivalent condition

$$\frac{7}{n+2} < \varepsilon \iff n > 7/\varepsilon - 2$$ continued...

There is also:

Proof by Contradiction

1. Start by assuming the opposite of the required result.

2. Work towards establishing a falsehood.

3. The final false statement means, by contradiction, that the original assumption is false $\implies$ required result is true.

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1. Start by assuming the opposite of the required result.
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Roughly 5%–10% of this subject's proofs are by contradiction.

INEQUALITIES AS BUILDING BLOCKS FOR REAL ANALYSIS PROOFS

QUICK QUIZ

Does

$$a < b \qquad \Longrightarrow \qquad a^2 < b^2 \quad ?$$

QUICK QUIZ

Does

$$a < b \qquad \Longrightarrow \qquad a^2 < b^2 \quad ?$$

ANSWER:

No, since $a = -1$ and $b = 0$ gives an easy counter-example.

$[-1 < 0$ but $(-1)^2 > 0^2]$

NEXT QUICK QUIZ

Does

$$0 < a < b \quad \Longrightarrow \quad a^2 < b^2 \;\; ?$$

NEXT QUICK QUIZ

Does

$$0 < a < b \quad \Longrightarrow \quad a^2 < b^2 \ ?$$

ANSWER:

Yes!

SECOND LAST QUICK QUIZ

Does

$$a > b \qquad \Longrightarrow \qquad \frac{1}{a} < \frac{1}{b} \quad ?$$

SECOND LAST QUICK QUIZ

Does

$$a > b \qquad \Longrightarrow \qquad \frac{1}{a} < \frac{1}{b} \quad ?$$

ANSWER:

No, since $a = 1$ and $b = -1$ gives an easy counter-example.

$[1 > -1 \quad \text{and} \quad 1/1 > 1/(-1)]$

LAST QUICK QUIZ

Does

$$0 < c < d \quad \Longrightarrow \quad \frac{1}{d} < \frac{1}{c} \; ?$$

LAST QUICK QUIZ

Does

$$0 < c < d \quad \Longrightarrow \quad \frac{1}{d} < \frac{1}{c} \; ?$$

ANSWER:

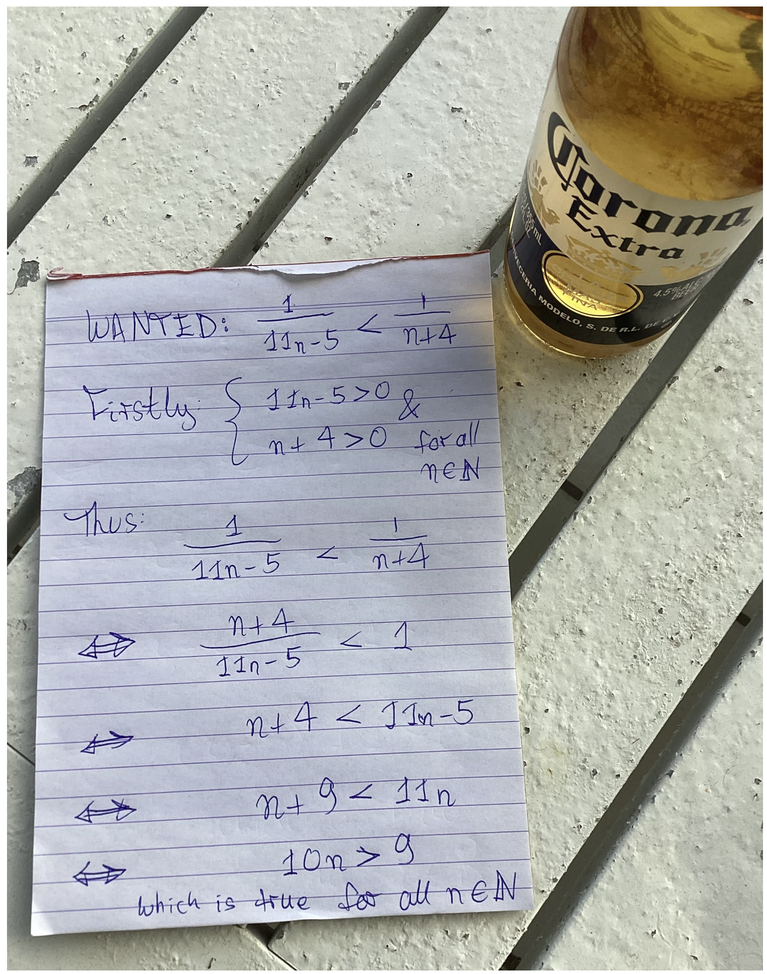
Yes!

Class Exercise

Prove that

$$\frac{1}{11n-5} < \frac{1}{n+4} \quad \text{for all } n \in \mathbb{N}.$$

Class Exercise Background Working (whilst at a pub)



Class Exercise Formal Proof

For all $n \in \mathbb{N}$

$$
\begin{aligned}
& 10n > 9 \\
\implies \quad & 11n > n + 9 \\
\implies \quad & 11n - 5 > n + 4 > 0 \\
\implies \quad & \frac{1}{11n - 5} < \frac{1}{n + 4}.
\end{aligned}
$$

Therefore,

$$\frac{1}{11n - 5} < \frac{1}{n + 4} \quad \text{for all } n \in \mathbb{N}.$$

Some Maths Problems ***Reprise***

1. Expand $(x+3)(4x^2-9x-6)$.
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Proving, from First Principles, that a Sequence Converges

Definition 1.16. A sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ is said to be convergent with limit x, if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n - x| < \epsilon$. We write $x_n \to x$. A sequence which does not converge is said to diverge.

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Closing Example

Using only the ε-N definition (Definition 1.16) prove that

$$\frac{1}{n+28} \to 0 \quad \text{or, equivalently,} \quad \lim_{n\to\infty}\left(\frac{1}{n+28}\right) = 0.$$

Formal Proof (Closing Example)

For each $\varepsilon > 0$ let

$N = \lceil 1/\varepsilon \rceil =$ the smallest integer greater than or equal to $1/\varepsilon$.

Then

$$n \geq N \quad \Longrightarrow \quad n \geq 1/\varepsilon \quad \Longrightarrow \quad n + 28 > 1/\varepsilon > 0.$$

Therefore

$$\frac{1}{n+28} < \varepsilon \quad \Longleftrightarrow \quad \left| \frac{1}{n+28} - 0 \right| < \varepsilon.$$

So from Definition 1.16,

$$\lim_{n \to \infty} \left(\frac{1}{n+28} \right) = 0.$$

Getting Ready for the Proof Parts of Quiz 1

- Review these slides.
- Familiarise yourself with Definition 1.16.
- Try Exercises B.1, B.2, B.3 and B.4.
- Attempt Practice Quiz 1.
- Get help from your tutor (Brian or Sam) at the Week 2 tutorial.
- Go through the solutions to be released at 4:00pm on Tuesday 4th August.

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