

UNIVERSITY OF TECHNOLOGY SYDNEY
School of Mathematical and Physical Sciences
35007 Real Analysis

Practice Quiz 2 – Solutions

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1. *Relevant page(s) in the notes: 19*

For each $n \in \mathbb{N}$ let

$$a_n = \frac{\sqrt{n^2 + 17} - \sqrt{n^2 + 12}}{8n + 3}.$$

Then, from the $(a - b)(a + b) = a^2 - b^2$ identity,

$$a_n = \frac{(\sqrt{n^2 + 17} - \sqrt{n^2 + 12})(\sqrt{n^2 + 17} + \sqrt{n^2 + 12})}{(8n + 3)(\sqrt{n^2 + 17} + \sqrt{n^2 + 12})} = \frac{5}{(8n + 3)(\sqrt{n^2 + 17} + \sqrt{n^2 + 12})}.$$

For each $n \in \mathbb{N}$, let

$$b_n = \frac{1}{n^2}.$$

From information given in the question

$$\sum_{n=1}^{\infty} b_n \text{ is convergent.}$$

For all $n \in \mathbb{N}$ note that $a_n > 0$, $b_n > 0$ and

$$\frac{a_n}{b_n} = \frac{5n^2}{(8n + 3)(\sqrt{n^2 + 17} + \sqrt{n^2 + 12})} = \frac{5}{(8 + 3/n)(\sqrt{1 + 17/n^2} + \sqrt{1 + 12/n^2})}.$$

This leads to

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{5}{16} \neq 0$$

and from the Limit Comparison Test (Theorem 1.40)

$$\sum_{n=1}^{\infty} a_n \text{ is also convergent.}$$

2. *Relevant page(s) in the notes: 27*

First note that for any $x, y \in [0, 2]$ we have

$$0 \leq x^2 + xy + y^2 \leq 12 \implies 4|x^2 + xy + y^2| \leq 48. \quad (1)$$

¹This document is less than a few months old which makes errors more likely. Updated versions with later dates may be released during the semester. Notifications will be sent via e-mail whenever an updated version is released.

For any given $\varepsilon > 0$ set $\delta = \varepsilon/48$. Then for each $x, y \in [0, 2]$,

$$|x - y| < \delta \iff |x - y| < \varepsilon/48.$$

Combining this result with the last inequality in (1) results in

$$4|x^2 + xy + y^2||x - y| < 48(\varepsilon/48) = \varepsilon.$$

This is equivalent to

$$|4x^3 - 4y^3| < \varepsilon \iff |f(x) - f(y)| < \varepsilon.$$

By Definition 2.13, $f : [0, 2] \rightarrow \mathbb{R}$ is uniformly continuous on $[0, 2]$.

Preliminary background working that led to an effective δ choice

For each $\varepsilon > 0$ need to get to

$$|4x^3 - 4y^3| < \varepsilon \iff 4|x^2 + xy + y^2||x - y| < \varepsilon.$$

Since

$$x, y \in [0, 2] \implies 0 \leq x^2 + xy + y^2 \leq 12 \implies 4|x^2 + xy + y^2| \leq 48$$

choosing $\delta = \varepsilon/48$ will allow the $|x - y| < \delta$ condition to be combined with the $4|x^2 + xy + y^2| \leq 48$ condition and get us to $4|x^2 + xy + y^2||x - y| < \varepsilon$.

3. Relevant page(s) in the notes: 25, 30

First note that for all $h \neq 0$

$$\frac{f(0 + h) - f(0)}{h} = \frac{|h|^3}{h} = \frac{h^2|h|}{h} = h|h|.$$

For the left limit with $h \rightarrow 0^-$ we have

$$h < 0 \implies |h| = -h \implies h|h| = -h^2.$$

Therefore

$$\lim_{h \rightarrow 0^-} \frac{f(0 + h) - f(0)}{h} = \lim_{h \rightarrow 0^-} (-h^2) = 0.$$

For the right limit with $h \rightarrow 0^+$ we have

$$h > 0 \implies |h| = h \implies h|h| = h^2.$$

Therefore

$$\lim_{h \rightarrow 0^+} \frac{f(0 + h) - f(0)}{h} = \lim_{h \rightarrow 0^+} h^2 = 0.$$

Since

$$\lim_{h \rightarrow 0^-} \frac{f(0 + h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{f(0 + h) - f(0)}{h} = 0,$$

f is differentiable at $x = 0$ and $f'(0) = 0$.

4. Relevant page(s) in the notes: 24

(a) For all $x \in \mathbb{R}$

$$\left| \frac{1}{x^2 + 11} - \frac{1}{27} \right| = \left| \frac{27 - (x^2 + 11)}{27(x^2 + 11)} \right| = \frac{|16 - x^2|}{27(x^2 + 11)} = \frac{|x - 4||x + 4|}{27(x^2 + 11)}.$$

(b) For each $\varepsilon > 0$ let $\delta = \min(1, 60\varepsilon)$. Then $|x - 4| < \delta$ implies

$$|x - 4| < 1 \quad \text{and} \quad |x - 4| < 60\varepsilon. \quad (2)$$

The first inequality in (2) implies

$$-1 < x - 4 < 1 \implies 3 < x < 5 \implies 7 < x + 4 < 9 \implies |x + 4| < 9. \quad (3)$$

Also note that

$$3 < x < 5 \implies 9 < x^2 < 25 \implies 20 < x^2 + 11 < 36 \implies \frac{1}{x^2 + 11} < \frac{1}{20}. \quad (4)$$

On combining the last inequalities in each of (2), (3) and (4) we obtain

$$\frac{|x - 4||x + 4|}{27(x^2 + 11)} < 60\varepsilon \times \frac{9}{27(20)} = \varepsilon.$$

From the part (a) result, this is equivalent to

$$\left| \frac{1}{x^2 + 11} - \frac{1}{27} \right| < \varepsilon.$$

Hence, by the ε - δ definition of a limit ((2) of Definition 2.1),

$$\lim_{x \rightarrow 4} \frac{1}{x^2 + 11} = \frac{1}{27}.$$

