

UNIVERSITY OF TECHNOLOGY SYDNEY
School of Mathematical and Physical Sciences
35007 Real Analysis

Practice Final Examination – Solutions

Date of this version: 14th October 2026¹

1. Relevant page(s) in the notes: 6, 21, 22

- (a) i. $\inf A = -\frac{1}{2}$ and $\sup A = 0$.
ii. $\inf A = 0$ and $\sup A = 16$.
- (b) i. $\liminf_{n \rightarrow \infty} a_n = -\frac{11}{4}$ and $\limsup_{n \rightarrow \infty} a_n = \frac{11}{4}$.
ii. $\liminf_{n \rightarrow \infty} a_n = \frac{1}{5}$ and $\limsup_{n \rightarrow \infty} a_n = \frac{1}{5}$.
iii. $\liminf_{n \rightarrow \infty} a_n = -\frac{3}{7}$ and $\limsup_{n \rightarrow \infty} a_n = \frac{1}{7}$.

2. Relevant page(s) in the notes: 18, 23, 68

First note that

$$f_n(x) = \sum_{k=1}^n g_k(x) \quad \text{where} \quad g_k(x) = \frac{(-1)^{2k+1} e^{x/k}}{(1 + 2e^{x/k})k\sqrt{k}}, \quad k = 1, \dots, n.$$

For all $k = 1, \dots, n$

$$|g_k(x)| = \left| \frac{(-1)^{2k+1} e^{x/k}}{(1 + 2e^{x/k})k\sqrt{k}} \right| = \frac{e^{x/k}}{(1 + 2e^{x/k})k\sqrt{k}} = \frac{1}{(e^{-x/k} + 2)k^{3/2}}$$

Next note that, for all $k \in \mathbb{N}$,

$$\begin{aligned} x \geq 0 &\implies x/k \geq 0 \implies e^{x/k} \geq 1 \implies e^{-x/k} \leq 1 \\ \implies e^{-x/k} + 2 &\leq 3 \implies \frac{1}{(e^{-x/k} + 2)k\sqrt{k}} \leq M_k \quad \text{where} \quad M_k = \frac{1}{k^{3/2}}. \end{aligned}$$

Since, for all $k \in \mathbb{N}$,

$$|g_k(x)| \leq M_k \quad \text{and} \quad \sum_{k=1}^{\infty} M_k \quad \text{is convergent}$$

by the Weierstrass M -test (Theorem 5.7), f_n is uniformly convergent on $[0, \infty)$.

¹This document is less than a few months old which makes errors more likely. Updated versions with later dates may be released during the semester. Notifications will be sent via e-mail whenever an updated version is released.

3. Relevant page(s) in the notes: 53

(a) Noting that

$$L(f, \mathcal{P}) = \frac{1}{n} \sum_{i=1}^n \frac{1}{i^2/n^2 + 9} \quad \text{and} \quad U(f, \mathcal{P}) = \frac{1}{n} \sum_{i=1}^n \frac{1}{(i-1)^2/n^2 + 9}$$

we then have

$$\begin{aligned} U(f, \mathcal{P}) - L(f, \mathcal{P}) &= \frac{1}{n} \left\{ \frac{1}{9} + \frac{1}{1/n^2 + 9} + \cdots + \frac{1}{(n-2)^2/n^2 + 9} + \frac{1}{(n-1)^2/n^2 + 9} \right. \\ &\quad \left. - \frac{1}{1/n^2 + 9} - \cdots - \frac{1}{(n-2)^2/n^2 + 9} - \frac{1}{(n-1)^2/n^2 + 9} - \frac{1}{1+9} \right\} \\ &= \frac{1}{n} \left(\frac{1}{9} - \frac{1}{10} \right) \\ &= \frac{1}{90n}. \end{aligned}$$

(b) For each $\varepsilon > 0$ consider any version of $\mathcal{P}$ for which $n > 1/(90\varepsilon)$. For any such n ,

$$90n\varepsilon > 1 \quad \implies \quad \frac{1}{90n} < \varepsilon.$$

On combining this result with that obtained in part (a) we have

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$$

which is Riemann's Criterion.

4. Relevant page(s) in the notes: 29, 37

We have

$$f(x) = 4x - 27.$$

Since f is a linear function of x , f is continuous and differentiable for all $x \in \mathbb{R}$. Also

$$f(3) = -15 \quad \text{and} \quad f(8) = 5 \quad \implies \quad f(3)f(8) = -75 < 0.$$

Therefore, by Theorem 2.19, there is a real number $c \in [3, 8]$ such that $f(c) = 0$. Hence, f has at least one root. Next note that

$$f'(x) = 4 > 0 \quad \text{for all} \quad x \in \mathbb{R}.$$

To prove that c is the unique root in $[3, 8]$ suppose that is another root d such that $d > c$. Then, by Rolle's Theorem (Theorem 3.9) there is a real number $r \in [c, d]$ such that $f'(r) = 0$. But this contradicts the above result that $f'(x) > 0$ for all $x \in \mathbb{R}$. A similar conclusion arises if d is another root such that $d < c$. Therefore c is the only root of f . In summary, $f(x) = 4x - 27$ has exactly one root in $[3, 8]$.

5. Relevant page(s) in the notes: 39, 40

(a) Both the numerator and denominator limits are 0, so by L'Hôpital's rule

$$\lim_{x \rightarrow 1} \frac{5x^4 - 4x^2 - 1}{10 - x - 9x^3} = \lim_{x \rightarrow 1} \frac{\frac{d}{dx}(5x^4 - 4x^2 - 1)}{\frac{d}{dx}(10 - x - 9x^3)} = \lim_{x \rightarrow 1} \frac{20x^3 - 8x}{(-1 - 27x^2)} = \frac{12}{-28} = -\frac{3}{7}.$$

(b) Let $f(x) = \ln(2x^6 + 8x^5)$ and $g(x) = \ln(5x^3 + 943)$. Since

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} g(x) = \infty$$

L'Hôpital's rule applies and we have

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)} \\ &= \lim_{x \rightarrow \infty} \frac{(12x^5 + 40x^4)/(2x^6 + 8x^5)}{(15x^2)/(5x^3 + 943)} \\ &= \lim_{x \rightarrow \infty} \frac{(12x^5 + 40x^4)(5x^3 + 943)}{(15x^2)(2x^6 + 8x^5)} \\ &= \lim_{x \rightarrow \infty} \frac{(12 + 40/x)(5 + 943/x^3)}{(15)(2 + 8/x)} \\ &= \frac{(12)(5)}{(15)(2)} \\ &= 2. \end{aligned}$$

(c) First note that for all $x > 0$

$$x^{1/x} = \exp\{(1/x) \ln(x)\} = \exp\left(\frac{\ln(x)}{x}\right).$$

Hence, using L'Hôpital's rule,

$$\lim_{x \rightarrow \infty} x^{1/x} = \exp\left(\lim_{x \rightarrow \infty} \frac{\ln(x)}{x}\right) = \exp\left(\lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln(x)}{\frac{d}{dx} x}\right) = \exp\left(\lim_{x \rightarrow \infty} \frac{1/x}{1}\right) = e^0 = 1.$$

6. Relevant page(s) in the notes: 6, 12, Supplementary Results

For $n \in \mathbb{N}$ and $x \in \mathbb{R}$ we have

$$\begin{aligned} |f_n(x) - f(x)| &= \left| \frac{x^2}{3 + 4x^2 + \{2 + (-1)^n\}/n} - \frac{x^2}{3 + 4x^2} \right| \\ &= \frac{x^2 |2 + (-1)^n|/n}{[3 + 4x^2 + \{2 + (-1)^n\}/n](3 + 4x^2)}. \end{aligned} \tag{1}$$

For all $n \in \mathbb{N}$ we have, by the triangle inequality,

$$|2 + (-1)^n| \leq |2| + |(-1)^n| = 3. \tag{2}$$

Also, noting that $2 + (-1)^n > 0$ for all $n \in \mathbb{N}$, we have

$$3 + 4x^2 + \{2 + (-1)^n\}/n > 3 + 4x^2 \quad \implies \quad \frac{1}{3 + 4x^2 + \{2 + (-1)^n\}/n} < \frac{1}{3 + 4x^2}. \quad (3)$$

On combining inequalities (1), (2) and (3) we have

$$|f_n(x) - f(x)| < \left(\frac{3}{n}\right) \left\{ \frac{x^2}{(3 + 4x^2)^2} \right\} \quad \text{for all } n \in \mathbb{N} \text{ and } x \in \mathbb{R}. \quad (4)$$

Then observe that for all $x \in \mathbb{R}$

$$3 + 3x^2 > 0 \quad \implies \quad 3 + 4x^2 > x^2 \quad \implies \quad \frac{x^2}{3 + 4x^2} < 1 \quad (5)$$

as well as

$$3 + 4x^2 \geq 3 \quad \implies \quad \frac{1}{3 + 4x^2} \leq \frac{1}{3}. \quad (6)$$

The inequalities in each of (4), (5) and (6) then imply that

$$|f_n(x) - f(x)| < \frac{1}{n} \quad \text{for all } n \in \mathbb{N} \text{ and } x \in \mathbb{R}.$$

Therefore, for all $n \in \mathbb{N}$,

$$0 \leq \sup\{|f_n(x) - f(x)| : x \in \mathbb{R}\} < \frac{1}{n}$$

and by the Sandwich Theorem (Theorem 1.23),

$$\lim_{n \rightarrow \infty} \sup\{|f_n(x) - f(x)| : x \in \mathbb{R}\} = 0.$$

By Theorem S.2,

$$f_n \text{ converges uniformly to } f \text{ on } \mathbb{R}.$$

Background working that led to the inequality in (5)

By inspection, it seems likely that

$$\frac{x^2}{3 + 4x^2} < 1 \quad \text{for all } x \in \mathbb{R}$$

so we will aim to get to this result. Note that

$$\frac{x^2}{3 + 4x^2} < 1 \quad \iff \quad x^2 < 3 + 4x^2 \quad \iff \quad 3x^2 + 3 > 0.$$

This suggests that we start with the assertion that $3x^2 + 3 > 0$ for all $x \in \mathbb{R}$.

