35007

More About Sequences

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A Reminder About the ...

SCREEN-FREE ZONE RULE

Today's Programme

1. More About Sequences
2. Series
3. ε-N Proof Examples (Whiteboard)

Classroom Rules

- NO switched on mobile phones.
- NO texting.
- NO *exting.
- NO open iPads.
- NO iPods.
- NO other iDevices.
- NO open laptops.

THEOREM S.1. *(The Ratio Test for Sequences) Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence such that* $x_n \neq 0$ *for all* $n \in \mathbb{N}$ *and*

$$L = \lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| \quad \text{exists.}$$

If $L < 1$ *then* $\lim_{n\to\infty} x_n = 0$. *If* $L > 1$ *then* x_n *diverges. Also, if* L *does not exist and*

$$\liminf_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = \infty \quad \text{then } x_n \text{ diverges.}$$

Ratio Test for Sequences Class Exercise

Determine whether or not x_n converges in each case.

1 $\quad x_n = (n+3)\left(\frac{8}{13}\right)^n, \quad n \in \mathbb{N}.$

2 $\quad x_n = \frac{938^n}{n!}, \quad n \in \mathbb{N}.$

★ 3 $\quad x_n = \frac{n^n}{n!}, \quad n \in \mathbb{N}.$

(★ = a bit more challenging.)

Solution to 1

Since

$$\left|\frac{x_{n+1}}{x_n}\right| = \frac{(n+4)(8/13)^{n+1}}{(n+3)(8/13)^n} = \left(\frac{1+4/n}{1+3/n}\right)\left(\frac{8}{13}\right)$$

we have

$$L = \left(\frac{8}{13}\right) \lim_{n\to\infty} \left(\frac{1+4/n}{1+3/n}\right) = \frac{8}{13} < 1.$$

By the Ratio Test for Sequences

$$\lim_{n\to\infty} (n+3)\left(\frac{8}{13}\right)^n = 0 \quad \Longrightarrow \quad x_n \text{ converges.}$$

Solution to 2

Since

$$\left|\frac{x_{n+1}}{x_n}\right| = \frac{938^{n+1}/(n+1)!}{938^n/n!} = \frac{938}{n+1} = \frac{938/n}{1+1/n}$$

we have

$$L = \lim_{n\to\infty} \left(\frac{938/n}{1+1/n}\right) = 0 < 1.$$

By the Ratio Test for Sequences

$$\lim_{n\to\infty} \frac{938^n}{n!} = 0 \quad \Longrightarrow \quad x_n \text{ converges.}$$

Solution to 3

Since

$$\left|\frac{x_{n+1}}{x_n}\right| = \frac{(n+1)^{n+1}/(n+1)!}{n^n/n!} = \left(1+\frac{1}{n}\right)^n$$

we have

$$L = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1.$$

By the Ratio Test for Sequences

$$x_n = \frac{n^n}{n!} \quad \text{diverges.}$$

Cauchy Sequences

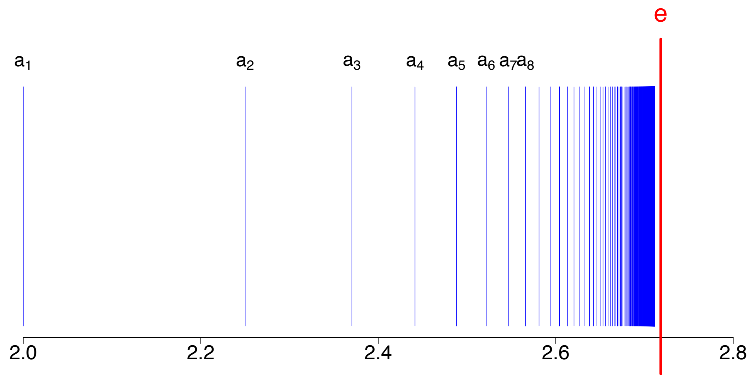
Cauchy Sequence Example

$$a_n = \left(1+\frac{1}{n}\right)^n \to e.$$

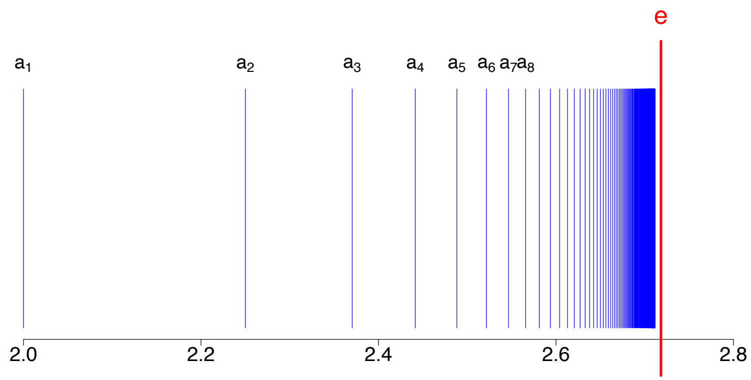
Cauchy Sequence Example

$$a_n = \left(1+\frac{1}{n}\right)^n \to e.$$

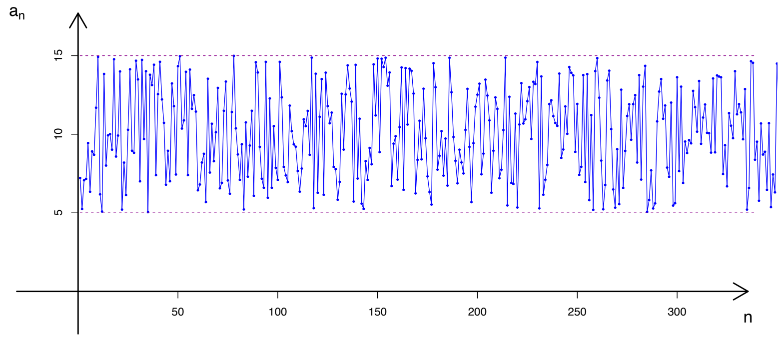
N	largest value of $\lvert a_m - a_n\rvert$ for all $m, n \geq N$
5	0.2300
10	0.1245
100	0.01347
500	0.0027

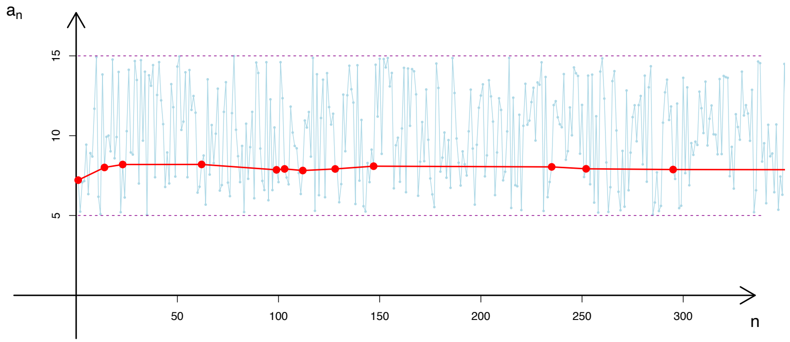


The Bolzano-Weierstrass Theorem



Cauchy's criterion illustration:** a_n **can converge if and only if the $|a_m - a_n|$ gaps get smaller as the sequence progresses. Intuitively, the sequence "bunches up".



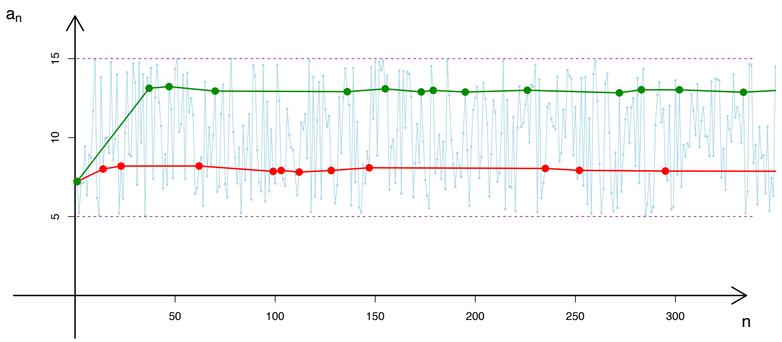


Class Exercise

Use the Bolzano-Weierstrass Theorem to prove that

$$b_n = \frac{(-1)^n(8 + 9/n)}{4 + 33/n}, \qquad n \in \mathbb{N},$$

has a convergent subsequence.



Class Exercise SOLUTION

We have

$$|b_n| = (8 + 9/n)\left(\frac{1}{4 + 33/n}\right). \qquad (1)$$

Firstly, for all $n \in \mathbb{N}$,

$$9/n \leq 9 \qquad \Longrightarrow \qquad 8 + 9/n \leq 8 + 9 = 17. \qquad (2)$$

Secondly, for all $n \in \mathbb{N}$,

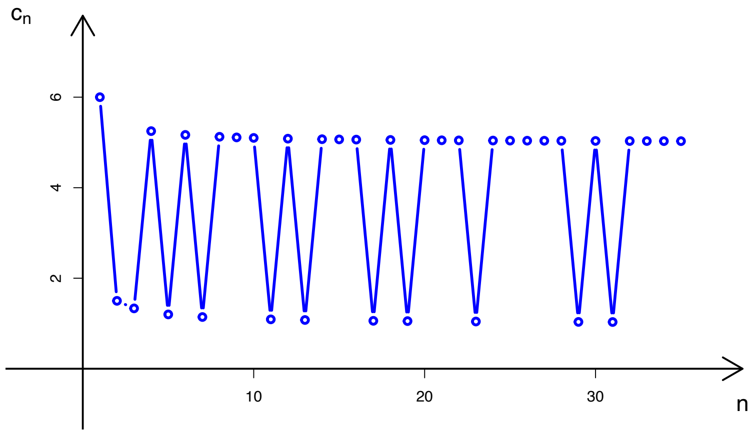
$$4 + 33/n > 4 \qquad \Longrightarrow \qquad \frac{1}{4 + 33/n} < \frac{1}{4}. \qquad (3)$$

Combining (1), (2) and (3) we obtain

$$|b_n| < \frac{17}{4} \quad \text{for all } n \in \mathbb{N} \quad \Longrightarrow \quad b_n \text{ is a bounded sequence.}$$

By the Bolzano-Weierstrass Theorem, b_n has a convergence subsequence.

lim inf and lim sup



Example

$$c_n = \begin{cases} 1 + 1/n & \text{if } n \text{ is a prime number,} \\ 5 + 1/n & \text{if } n \text{ is a composite number or } n = 1. \end{cases}$$

Verbal Definitions of $\liminf$, $\limsup$

$\liminf_{n\to\infty} c_n$ is

the lowest limit we can get using subsequences of c_n.

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$\liminf_{n\to\infty} c_n$ is

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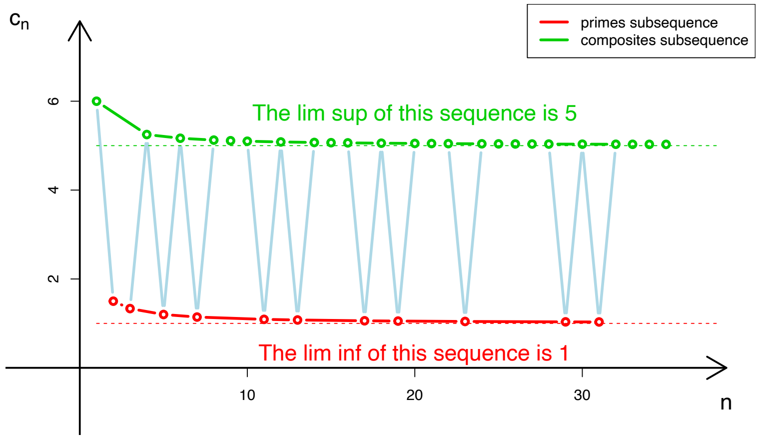
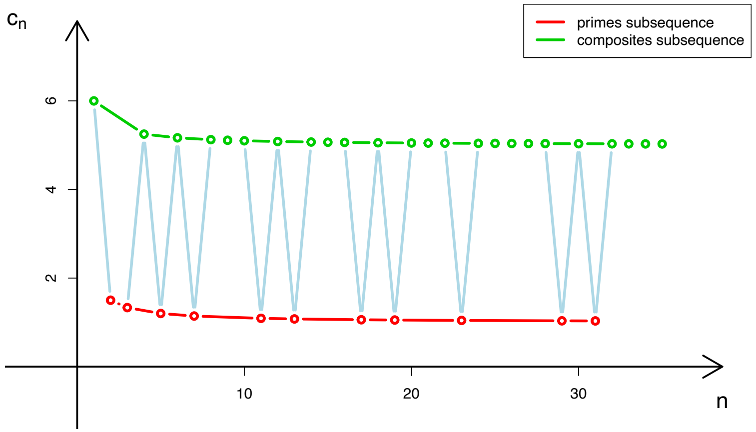
$\limsup_{n\to\infty} c_n$ is

the highest limit we can get using subsequences of c_n.





Class Exercise

For the sequence from the previous class exercise,

$$b_n = \frac{(-1)^n(8+9/n)}{4+33/n}, \quad n \in \mathbb{N},$$

write down

$$\liminf_{n\to\infty} b_n \quad \text{and} \quad \limsup_{n\to\infty} b_n.$$

Class Exercise SOLUTION

$$\liminf_{n\to\infty} b_n = \liminf_{n\to\infty} \frac{(-1)^n(8+9/n)}{4+33/n} = -2$$

(based on the n odd subsequence)

$$\limsup_{n\to\infty} b_n = \limsup_{n\to\infty} \frac{(-1)^n(8+9/n)}{4+33/n} = 2$$

(based on the n even subsequence)

The Most Relevant Exercises
(from the subject exercise sets)

Exercise Set B

Exercise Set C

♦ ♦ ♦