35007

CLASS 4 ; LIVE ε-δ PROOF EXAMPLE

TO PROVE: $\lim\limits_{x \to 1} \dfrac{x}{x+1} = \frac{1}{2}$

BACKGROUND WORKING TO OBTAIN δ

For $\varepsilon > 0$ we need to get to

$$\left| \frac{x}{x+1} - \frac{1}{2} \right| < \varepsilon \iff \left| \frac{2x - (1+x)}{2(1+x)} \right| < \varepsilon$$

$$\iff |x-1| \left(\frac{1}{2|x+1|} \right) < \varepsilon \quad \ldots (1)$$

Note that

$$|x-1| < \delta \iff -\delta < x-1 < \delta$$

$$\iff 2-\delta < x+1 < 2+\delta.$$

To keep $x+1$ away from zero we will start with $0 < \delta \leq 1$. Under this restriction

$$|x-1| < \delta \Rightarrow |x-1| < 1$$

$$\iff -1 < x-1 < 1$$

$$\Leftrightarrow \quad 1 < x+1 < 3$$

$$\Leftrightarrow \quad \frac{1}{3} < \frac{1}{x+1} < 1$$

$$\Leftrightarrow \quad \frac{1}{6} < \frac{1}{2|x+1|} < \frac{1}{2} \quad \ldots\ldots (2)$$

Staring at (1) and (2), a way to get to (1) is having

$$|x-1| < 2\varepsilon \quad \text{and} \quad \frac{1}{2|x+1|} < \frac{1}{2}$$

Since then we would have

$$|x-1|\left(\frac{1}{2|x+1|}\right) < 2\varepsilon \times \frac{1}{2} = \varepsilon.$$

Hence, we want:

1. x to be within a radius of length 1 centred on 1.
2. x to be within a radius of length 2ε centred on 1.

$\rightarrow$ $\delta = \min(2\varepsilon, 1)$ should suffice.

FORMAL PROOF

NOTE: Part (1) of Definition 2·1, which is relatively easy, is being assumed here.

For any given $\varepsilon > 0$ let $\delta = \min(1, 2\varepsilon)$.

Then

$$|x-1| < \delta \Rightarrow |x-1| < 1 \text{ and } |x-1| < 2\varepsilon \quad \ldots (1)$$

Next note that

$$|x-1| < 1 \Rightarrow 1 < x+1 < 3$$

$$\Rightarrow \quad \frac{1}{2|x+1|} < \frac{1}{2} \quad \ldots\ldots (2)$$

Combining the last inequality of (1) with (2) leads to

$$|x-1| \left(\frac{1}{2|x+1|}\right) < (2\varepsilon)(\tfrac{1}{2}) = \varepsilon.$$

Simple algebra leads to

$$|x-1| \left(\frac{1}{2|x+1|}\right) = \left|\frac{x}{x+1} - \frac{1}{2}\right| \text{ for all } x \in \mathbb{R}.$$

Hence $|x-1| < \delta \Rightarrow \left|\frac{x}{x+1} - \frac{1}{2}\right| < \varepsilon$. By (2) of Definition 2.1, $\lim_{x \to 1}\left(\frac{x}{x+1}\right) = 1/2$.