35007

Limits for Functions

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Differences Compared with Sequence Limits

Opening Examples

1. $\lim\limits_{x\to 3} x^2 = 9$
2. $\lim\limits_{x\to 0} \dfrac{\sin(x)}{x} = 1$
3. $\lim\limits_{x\to\infty} \dfrac{\ln(x)}{\ln(x^3+x)} = \dfrac{1}{3}$

How This Segment Differs:

A. We are now dealing with functions $f:\mathbb{R}\to\mathbb{R}$.

B. $n\to\infty$ replaced by $x\to c$ for general $c\in\mathbb{R}$ (or $\pm\infty$).

ε-δ Proofs

ε - δ Proofs

- ***The hardest part of the entire subject.***
- ***Learning by doing – see Exercise Set E.***
- ***The following slides show useful facts and steps.***

Power Difference Identities

For all $a, b \in \mathbb{R}$:

$$a^2 - b^2 = (a-b)(a+b)$$

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

$$a^4 - b^4 = (a-b)(a+b)(a^2+b^2)$$

The Triangle Inequality

For all $a, b \in \mathbb{R}$:

$$|a+b| \le |a| + |b|.$$

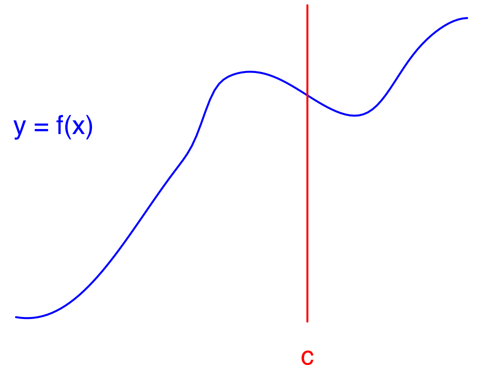
[Lemma 1.10 of the subject notes.]

The Reverse Triangle Inequality

For all $a, b \in \mathbb{R}$:

$$|a - b| \geq |a| - |b|.$$

[Lemma 1.28 of the subject notes.]



A well-behaved function $f : \mathbb{R} \to \mathbb{R}$ near $x = c$.

Strategy for Well-Behaved Functions (No Asymptotes)

Definition

$$\lim_{x \to c} f(x) = L$$

if and only if

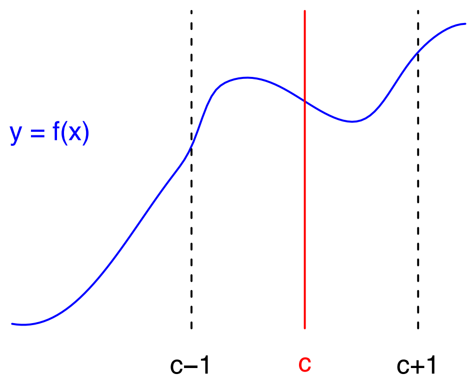
for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x \in \mathbb{R},\ |x - c| < \delta \quad \Longrightarrow \quad |f(x) - L| < \varepsilon.$$

PHASE 1:

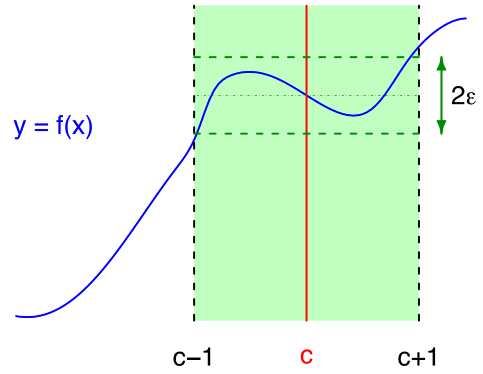
Forcing

$$\delta \le 1.$$



Force $\delta \le 1$. Then

$$|x-c| < \delta \implies c-1 < x < c+1.$$



Now study $f(x)$ for x only in the shaded region.
Refine the δ choice (Phase 2) to achieve the

$$|f(x)-f(c)| < \varepsilon \quad \text{goal.}$$

Example

TO PROVE:

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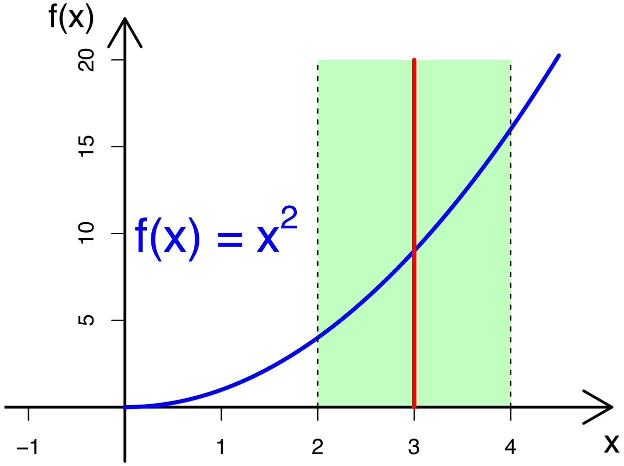
Under ***Phase 1*** with $\delta \leq 1$ we have

$$|x-3| < \delta \implies |x-3| < 1 \implies 2 < x < 4.$$

PHASE 2:

Refining δ for arbitrary $\varepsilon > 0.$



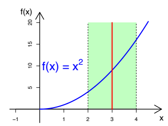
We need:

$$|x^2 - 9| < \varepsilon \iff |x-3||x+3| < \varepsilon. \quad \star \qquad (1)$$

Under our Phase 1 $\delta \leq 1$ restriction

$$|x+3| = |x-3+6| \leq |x-3| + 6 < 1 + 6 = 7.$$

That is, $|x+3| < 7\star$ within this shaded green region.



Staring at $\star$ and $\star$ it becomes apparent that taking

$$|x-3| < \frac{\varepsilon}{7} \implies |x-3||x+3| < \frac{\varepsilon}{7} \times 7 = \varepsilon.$$

Getting to the Final δ

Phase 1 gave us $|x-3| < 1$.

Phase 2 gave us $|x-3| < \frac{\varepsilon}{7}$.

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Verbally, we want

1. x to be within a radius of length 1 centred on $c = 3$.
2. x to be within a radius of length $\varepsilon/7$ centred on $c = 3$.

$$\longrightarrow \quad \delta = \min\left(1, \frac{\varepsilon}{7}\right) \quad \text{will be good enough.}$$

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Phase 2 gave us $|x-3| < \frac{\varepsilon}{7}$.

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Formal Proof

For any given $\varepsilon > 0$ set $\delta = \min\left(1, \varepsilon/7\right)$. Then $|x-3| < \delta$ implies that

$$|x-3| < \frac{\varepsilon}{7} \quad \text{and} \quad |x-3| < 1. \qquad \star$$

Then, using the first inequality in ★ and the triangle inequality

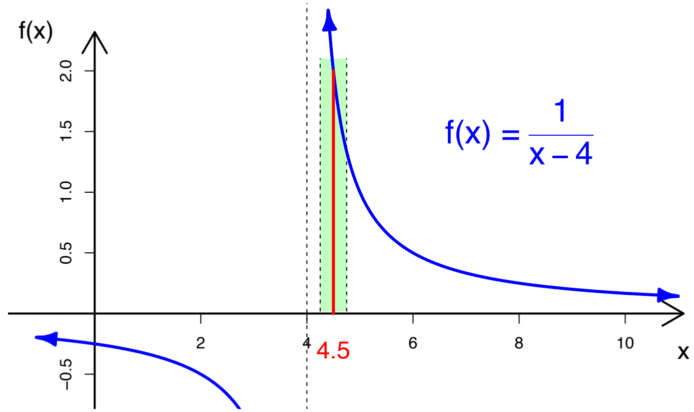
$$|x^2-9| = |x^2-3^2| = |x+3||x-3| < \frac{\varepsilon|x+3|}{7} = \frac{\varepsilon|x-3+6|}{7} \leq \frac{\varepsilon(|x-3|+6)}{7}.$$

Combining this result with the second inequality of ★ we obtain

$$|x^2-9| < \frac{\varepsilon(1+6)}{7} = \varepsilon.$$

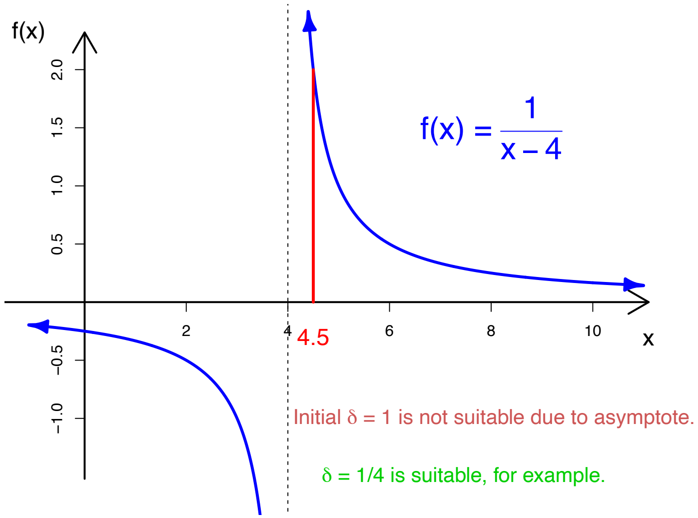
Hence, by the ε-δ definition of a limit ((2) of Definition 2.1), $\lim_{x \to 3} x^2 = 9$.

Less Well-Behaved Functions



Phase 1 should have, for example, $\delta \le 1/4$

$$\implies \quad |x - 4.5| < \frac{1}{4}.$$



The Most Relevant Exercises
(from the subject exercise sets)

E.1 – E.4, E.5 – E.9.

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