

35007

Limits for Functions

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Opening Examples

1. $\lim_{x \rightarrow 3} x^2 = 9$

2. $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$

3. $\lim_{x \rightarrow \infty} \frac{\ln(x)}{\ln(x^3 + x)} = \frac{1}{3}$

*Differences
Compared with
Sequence
Limits*

How This Segment Differs:

A. We are now dealing with functions $f : \mathbb{R} \rightarrow \mathbb{R}$.

B. $n \rightarrow \infty$ replaced by $x \rightarrow c$ for general $c \in \mathbb{R}$ (or $\pm\infty$).

ε - δ

Proofs

ε - δ Proofs

- *The **hardest part** of the entire subject.*
- *Learning by doing
– see Exercise Set E.*
- *The following slides show useful **facts** and **steps**.*

Power Difference Identities

For all $a, b \in \mathbb{R}$:

$$a^2 - b^2 = (a - b)(a + b)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

$$a^4 - b^4 = (a - b)(a + b)(a^2 + b^2)$$

The Triangle Inequality

For all $a, b \in \mathbb{R}$:

$$|a + b| \leq |a| + |b|.$$

[Lemma 1.10 of the subject notes.]

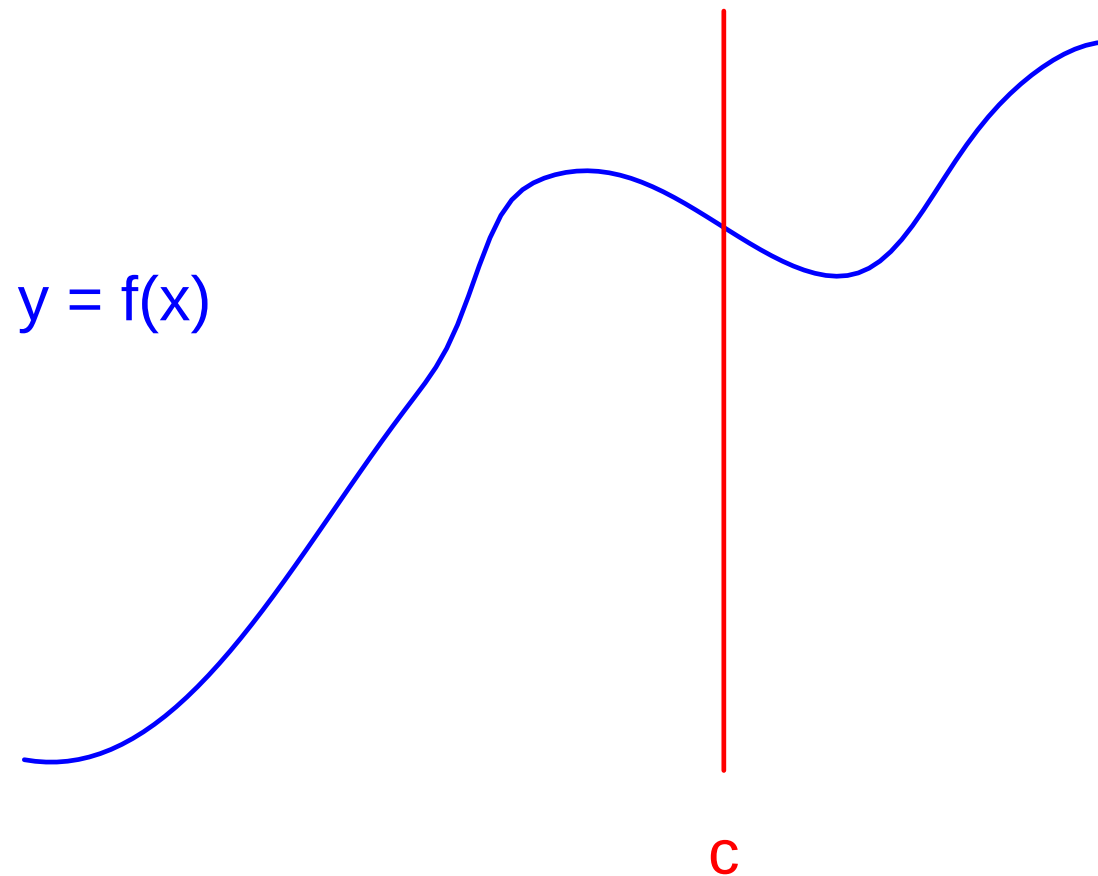
*The **Reverse** Triangle Inequality*

For all $a, b \in \mathbb{R}$:

$$|a - b| \geq |a| - |b|.$$

[Lemma 1.28 of the subject notes.]

Strategy for
Well-Behaved
Functions
(No Asymptotes)



A well-behaved function $f : \mathbb{R} \rightarrow \mathbb{R}$ near $x = c$.

Definition

$$\lim_{x \rightarrow c} f(x) = L$$

if and only if

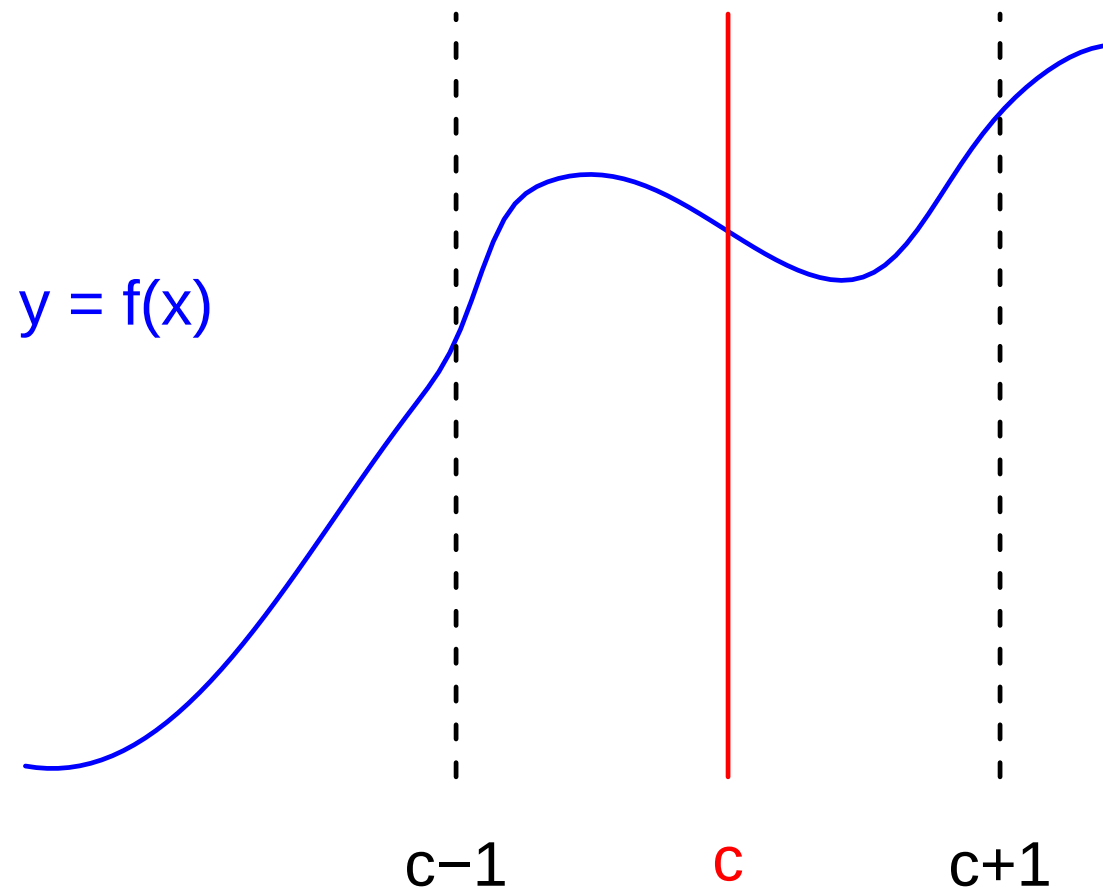
for each $\varepsilon > 0$, there exists $\delta > 0$
such that

$$x \in \mathbb{R}, \quad |x - c| < \delta \quad \implies \quad |f(x) - L| < \varepsilon.$$

PHASE 1:

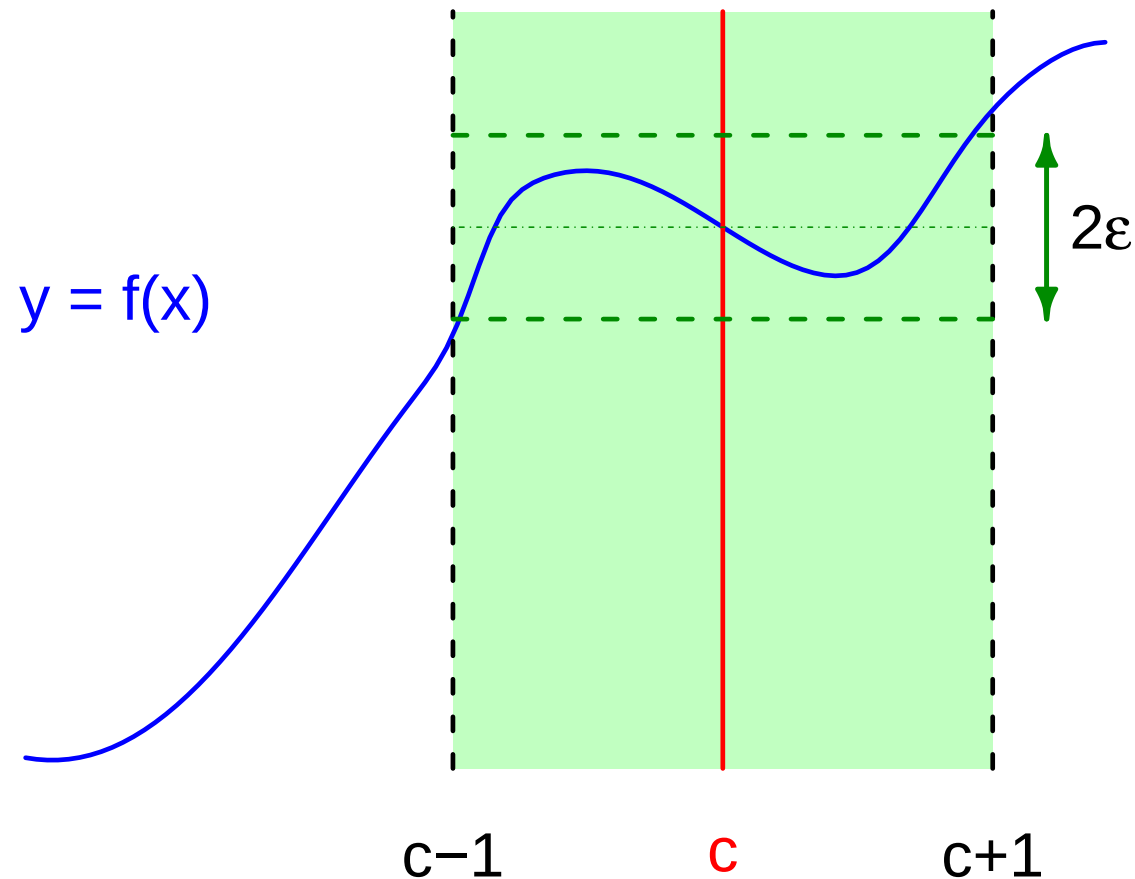
Forcing

$$\delta \leq 1.$$



Force $\delta \leq 1$. Then

$$|x - c| < \delta \implies c - 1 < x < c + 1.$$



Now study $f(x)$ for x only in the shaded region.
Refine the δ choice (Phase 2) to achieve the

$$|f(x) - f(c)| < \varepsilon \quad \text{goal.}$$

Example

TO PROVE:

$$\lim_{x \rightarrow 3} x^2 = 9.$$

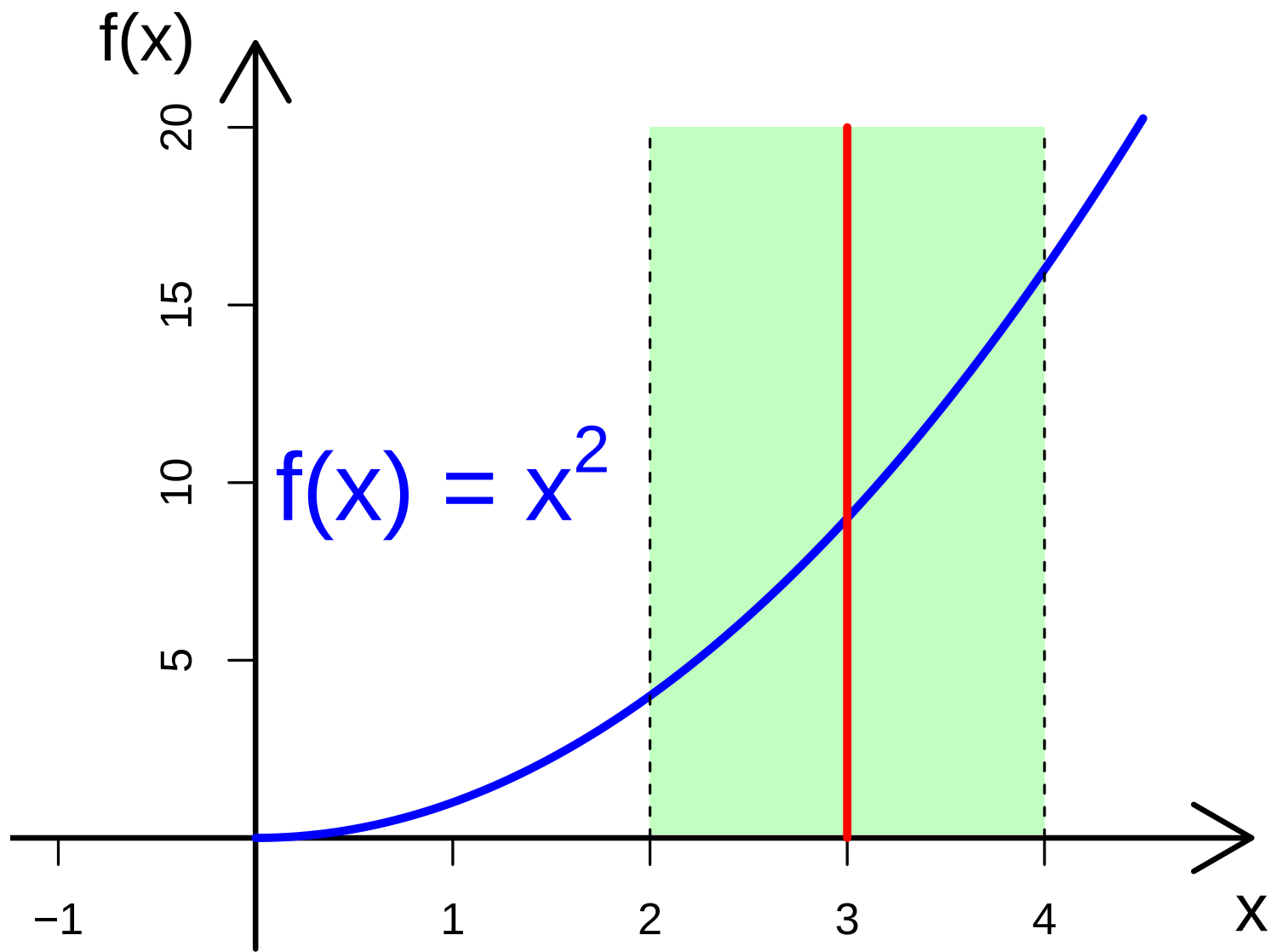
Example

TO PROVE:

$$\lim_{x \rightarrow 3} x^2 = 9.$$

Under ***Phase 1*** with $\delta \leq 1$ we have

$$|x - 3| < \delta \implies |x - 3| < 1 \implies 2 < x < 4.$$



PHASE 2:

***Refining δ for
arbitrary $\varepsilon > 0$.***

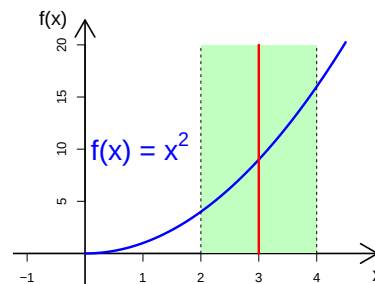
We need:

$$|x^2 - 9| < \varepsilon \iff |x - 3||x + 3| < \varepsilon. \quad \star \quad (1)$$

Under our Phase 1 $\delta \leq 1$ restriction

$$|x + 3| = |x - 3 + 6| \leq |x - 3| + 6 < 1 + 6 = 7.$$

That is, $|x + 3| < 7 \star$ within this shaded green region.



Staring at $\star$ and $\star$ it becomes apparent that taking

$$|x - 3| < \frac{\varepsilon}{7} \implies |x - 3||x + 3| < \frac{\varepsilon}{7} \times 7 = \varepsilon.$$

Getting to the Final δ

Phase 1 gave us $|x - 3| < 1$.

Phase 2 gave us $|x - 3| < \frac{\varepsilon}{7}$.

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Verbally, we want

1. x to be within a radius of length 1 centred on $c = 3$.
2. x to be within a radius of length $\varepsilon/7$ centred on $c = 3$.

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Phase 2 gave us $|x - 3| < \frac{\varepsilon}{7}$.

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1. x to be within a radius of length 1 centred on $c = 3$.
2. x to be within a radius of length $\varepsilon/7$ centred on $c = 3$.

$\longrightarrow \delta = \min \left(1, \frac{\varepsilon}{7} \right)$ will be good enough.

Formal Proof

For any given $\varepsilon > 0$ set $\delta = \min(1, \varepsilon/7)$. Then $|x - 3| < \delta$ implies that

$$|x - 3| < \frac{\varepsilon}{7} \quad \text{and} \quad |x - 3| < 1. \quad \star$$

Then, using the first inequality in $\star$ and the triangle inequality

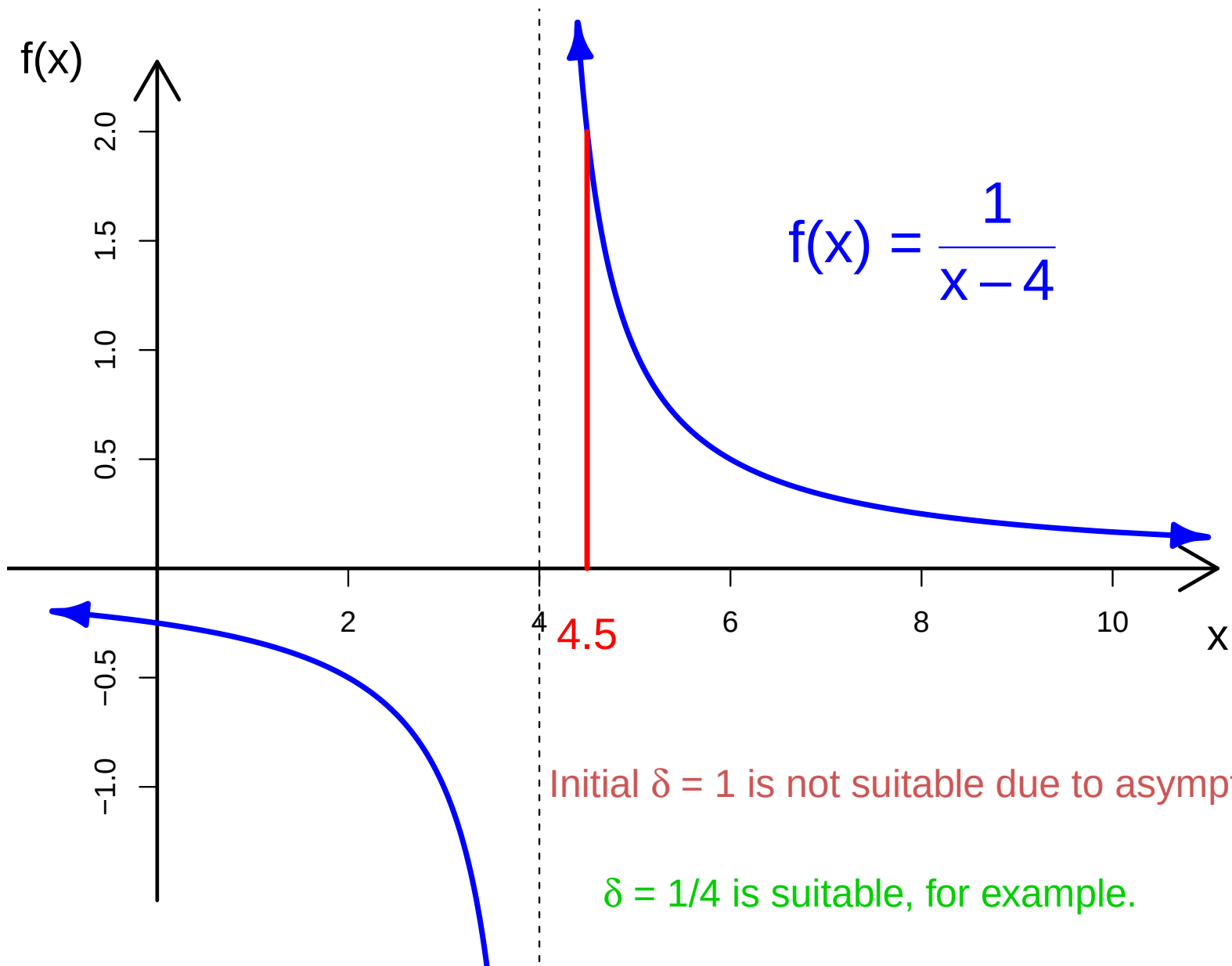
$$|x^2 - 9| = |x^2 - 3^2| = |x + 3||x - 3| < \frac{\varepsilon|x + 3|}{7} = \frac{\varepsilon|x - 3 + 6|}{7} \leq \frac{\varepsilon(|x - 3| + 6)}{7}.$$

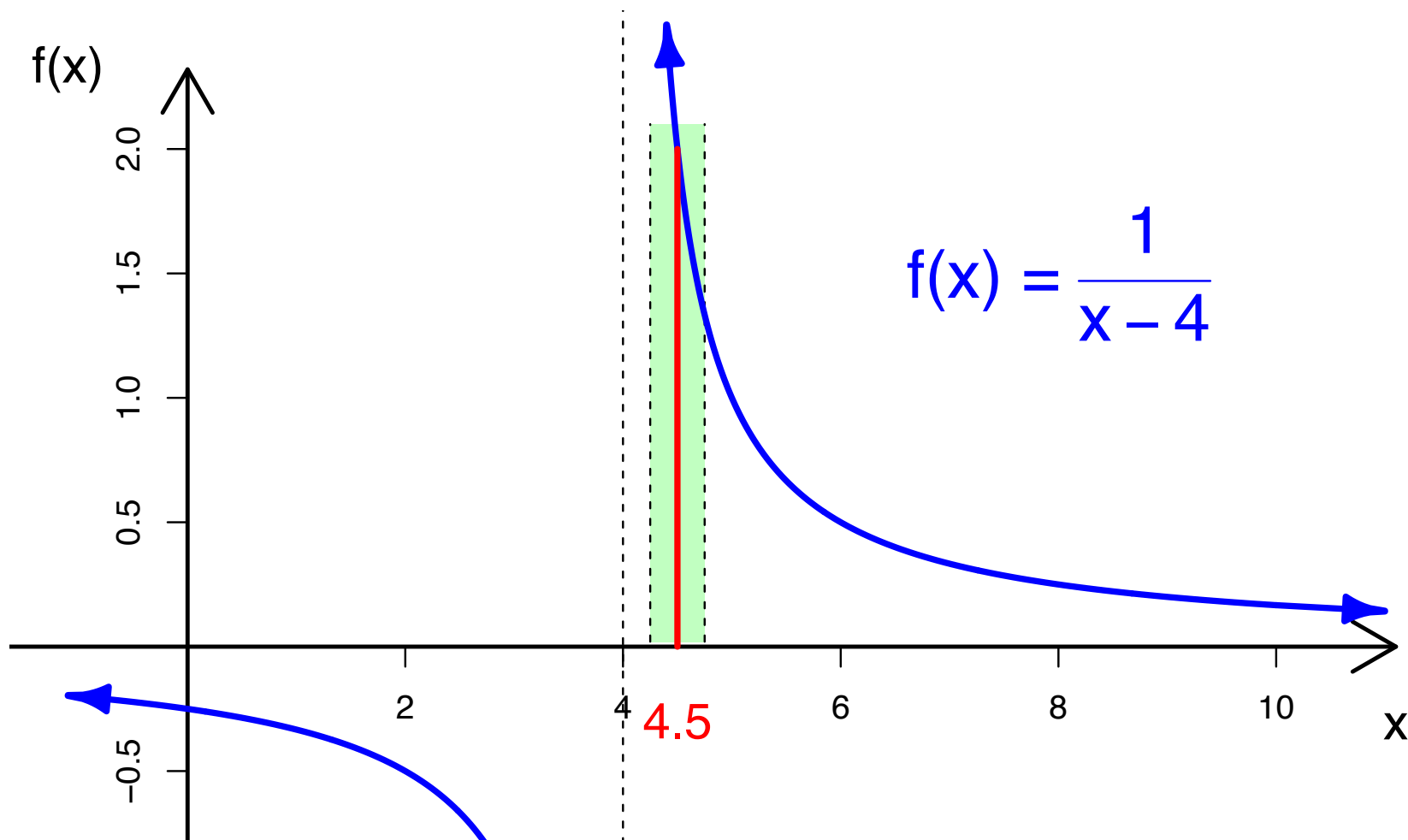
Combining this result with the second inequality of $\star$ we obtain

$$|x^2 - 9| < \frac{\varepsilon(1 + 6)}{7} = \varepsilon.$$

Hence, by the ε - δ definition of a limit ((2) of Definition 2.1), $\lim_{x \rightarrow 3} x^2 = 9$.

*Less
Well-Behaved
Functions*





Phase 1 should have, for example, $\delta \leq 1/4$

$$\implies |x - 4.5| < \frac{1}{4}.$$

The Most Relevant Exercises
(from the subject exercise sets)

E.1 – E.4, E.5 – E.9.

