

35007

# Solving Integrals Using Series

Professor Matt Wand

University of Technology Sydney

WANTED:

$$I = \int_0^{\infty} \frac{x^3}{e^x - 1} dx$$

Possibly useful first steps:  $\int_0^{\infty} \frac{x^3}{e^x - 1} dx$

For all  $x > 0$ , the geometric series result

$$\frac{1}{1-r} = 1 + r + r^2 + \dots, \quad |r| < 1, \text{ gives}$$

$$\begin{aligned} \frac{1}{e^x - 1} &= e^{-x} \left( \frac{1}{1 - e^{-x}} \right) = e^{-x} \sum_{k=0}^{\infty} (e^{-x})^k \\ &= e^{-x} \lim_{n \rightarrow \infty} \sum_{k=0}^n (e^{-x})^k. \end{aligned}$$

Continuing the quest to solve  $\int_0^{\infty} \frac{x^3}{e^x - 1} dx$

From the previous slide

$$\frac{x^3}{e^x - 1} = x^3 e^{-x} \lim_{n \rightarrow \infty} \sum_{k=0}^n (e^{-x})^k, \quad x > 0.$$

Therefore

$$I = \int_0^{\infty} \lim_{n \rightarrow \infty} f_n(x) dx$$

where

$$f_n(x) = x^3 \sum_{k=0}^n e^{-(k+1)x}.$$

### Proving Monotonicity of

$$f_n(x) = x^3 \sum_{k=0}^n e^{-(k+1)x}$$

For all  $x > 0$  and  $n \in \mathbb{N}$

$$\begin{aligned} f_{n+1}(x) - f_n(x) &= \sum_{k=0}^{n+1} e^{-(k+1)x} - \sum_{k=0}^n e^{-(k+1)x} \\ &= e^{-(n+2)x} > 0. \end{aligned}$$

Hence

$$f_{n+1}(x) > f_n(x).$$

### The Story So Far...

$$I = \int_0^\infty \lim_{n \rightarrow \infty} f_n(x) dx \quad \text{where}$$

$$f_{n+1}(x) > f_n(x) \quad \text{on} \quad (0, \infty).$$

***Tricky aspects  
with respect to  
limit / integral  
interchange***

### The Story So Far...

$$I = \int_0^\infty \lim_{n \rightarrow \infty} f_n(x) dx \quad \text{where}$$

$$f_{n+1}(x) > f_n(x) \quad \text{on} \quad (0, \infty).$$

***If, instead, we were dealing with***

$$I = \int_a^b \lim_{n \rightarrow \infty} f_n(x) dx \quad \text{where}$$

$$f_{n+1}(x) > f_n(x) \quad \text{on} \quad [a, b]$$

***then we could justify***  $I = \lim_{n \rightarrow \infty} \int_a^b f_n(x) dx.$

## The Story So Far...

$$I = \int_0^\infty \lim_{n \rightarrow \infty} f_n(x) dx \quad \text{where}$$

$$f_{n+1}(x) > f_n(x) \quad \text{on} \quad (0, \infty).$$

*If, instead, we were dealing with*

$$I = \int_a^b \lim_{n \rightarrow \infty} f_n(x) dx \quad \text{where}$$

$$f_{n+1}(x) > f_n(x) \quad \text{on} \quad [a, b]$$

*then we could justify*  $I = \lim_{n \rightarrow \infty} \int_a^b f_n(x) dx$ .

*For now we will proceed with  $(0, \infty)$  treated like  $[a, b]$  ( $a$  and  $b$  finite), and discuss justification later.*

*Next steps after limit / integral interchange  
(assumed justifiable for now)*

$$\begin{aligned} I &= \int_0^\infty \lim_{n \rightarrow \infty} f_n(x) dx \\ &= \lim_{n \rightarrow \infty} \int_0^\infty f_n(x) dx \\ &= \lim_{n \rightarrow \infty} \int_0^\infty x^3 \sum_{k=0}^n e^{-(k+1)x} dx \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^n \int_0^\infty x^3 e^{-(k+1)x} dx \\ &= \sum_{k=0}^\infty \int_0^\infty x^3 e^{-(k+1)x} dx \end{aligned}$$

## The Story So Far...

$$I = \sum_{k=0}^\infty \int_0^\infty x^3 e^{-(k+1)x} dx$$

## The Story So Far...

$$I = \sum_{k=0}^\infty \int_0^\infty x^3 e^{-(k+1)x} dx$$

*Substitution time:*

$$\begin{aligned} u = (k+1)x &\implies x = \frac{u}{k+1} \implies dx = \frac{du}{k+1}. \\ x = 0 &\implies u = 0 \quad \text{and} \quad x = \infty \implies u = \infty. \end{aligned}$$

## The Story So Far...

$$I = \sum_{k=0}^{\infty} \int_0^{\infty} x^3 e^{-(k+1)x} dx$$

**Substitution time:**

$$u = (k+1)x \implies x = \frac{u}{k+1} \implies dx = \frac{du}{k+1}.$$

$$x = 0 \implies u = 0 \quad \text{and} \quad x = \infty \implies u = \infty.$$

$$\longrightarrow I = \sum_{k=0}^{\infty} \frac{1}{(k+1)^4} \int_0^{\infty} u^3 e^{-u} du = 3! \sum_{k=0}^{\infty} \frac{1}{(k+1)^4}$$

## Final Steps...

Letting  $j = k + 1$ :

$$I = 6 \sum_{k=0}^{\infty} \frac{1}{(k+1)^4} = 6 \sum_{j=1}^{\infty} \frac{1}{j^4}.$$

## Final Steps...

Letting  $j = k + 1$ :

$$I = 6 \sum_{k=0}^{\infty} \frac{1}{(k+1)^4} = 6 \sum_{j=1}^{\infty} \frac{1}{j^4}.$$

From [equation \(1.2\) on page 18](#) of the notes

$$\sum_{j=1}^{\infty} \frac{1}{j^4} = \frac{\pi^4}{90}.$$

## Final Steps...

Letting  $j = k + 1$ :

$$I = 6 \sum_{k=0}^{\infty} \frac{1}{(k+1)^4} = 6 \sum_{j=1}^{\infty} \frac{1}{j^4}.$$

From [equation \(1.2\) on page 18](#) of the notes

$$\sum_{j=1}^{\infty} \frac{1}{j^4} = \frac{\pi^4}{90}.$$

$$\longrightarrow I = \int_0^1 \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}.$$

***However, we  
still have not justified  
the limit / integral  
interchange due to  
(0,∞) versus [a,b].***

***Using the Weierstrass M-test to get  
uniform convergence over (0, ∞).***

$$f_n(x) = \sum_{k=0}^n x^3 e^{-(k+1)x}, \quad x > 0.$$

Several lines of analysis (omitted here) leads to

$$\left| x^3 e^{-(k+1)x} \right| \leq \frac{27}{e^3(k+1)^3} \text{ for all } x > 0 \text{ and } k \in \mathbb{N}.$$

Since

$$\sum_{k=0}^{\infty} \frac{27}{e^3(k+1)^3} = \frac{27}{e^3} \sum_{k=1}^{\infty} \frac{1}{k^3} \text{ converges,}$$

by the **Weierstrass M-test**  $f_n$  converges uniformly to  
 $f(x) = \frac{x^3}{e^x - 1}$  over  $(0, \infty)$ .

We now have

$f_n$  converges uniformly to  $f$  over  $(0, \infty)$ .

and want to **FULLY JUSTIFY**

$$\lim_{n \rightarrow \infty} \int_0^{\infty} f_n(x) dx = \int_0^{\infty} \lim_{n \rightarrow \infty} f_n(x) dx.$$

We now have

$f_n$  converges uniformly to  $f$  over  $(0, \infty)$ .

and want to **FULLY JUSTIFY**

$$\lim_{n \rightarrow \infty} \int_0^{\infty} f_n(x) dx = \int_0^{\infty} \lim_{n \rightarrow \infty} f_n(x) dx.$$

However **Theorem 5.10** with  $[a, b]$  a **bounded** interval does not cover the  $(0, \infty)$  case. **Uniform convergence over  $(0, \infty)$  is not enough** and **advanced literature** says we **also** need to prove a condition such as

$$\lim_{b \rightarrow \infty} \sup \left\{ \int_{1/b}^b |f_n(x)| dx : n \in \mathbb{N} \right\} = 0.$$

## Notes and Exercise Examples

$$\int_0^1 \frac{\ln(x)}{x+1} dx = -\frac{\pi^2}{12} \quad \text{pages 91–92}$$

$$\int_0^\infty \frac{x}{e^x(e^x+1)} dx = 1 - \frac{\pi^2}{12} \quad \text{page 92}$$

$$\int_0^1 \frac{\tan^{-1}(x)}{x} dx = G \quad \text{[Catalan's constant] page 93}$$

$$\int_0^\infty \frac{\ln(x)}{1+x^2} dx = G \quad \text{Exercise L.4.}$$

$$\int_0^1 \frac{x \ln^3(x)}{1-x^4} dx = -\frac{\pi^4}{256} \quad \text{Exercise L.5.}$$

$$\int_0^1 \frac{x \ln^5(x)}{x+1} dx = -\frac{31\pi^6}{352} \quad \text{Exercise L.6.}$$

$$\int_0^\infty \frac{x}{e^x(e^x+1)} dx = 1 - \frac{\pi^2}{12} \quad \text{example in subject notes}$$

Example 6.12. Now let us use the same method to evaluate

$$I = \int_0^\infty \frac{x e^{-x}}{e^x + 1} dx. \quad (6.36)$$

To use a geometric series we write

$$\frac{1}{e^x + 1} = \frac{1}{e^x(1 + e^{-x})} = e^{-x} (1 - e^{-x} + e^{-2x} - e^{-3x} + \dots). \quad (6.37)$$

This converges because  $|e^{-x}| < 1$  for  $x > 0$ . Hence

$$\begin{aligned} I &= \int_0^\infty x e^{-2x} \sum_{n=0}^\infty (-1)^n e^{-nx} dx \\ &= \sum_{n=0}^\infty (-1)^n \int_0^\infty x e^{-(n+2)x} dx \\ &= \sum_{n=0}^\infty (-1)^n \left( \left[ \frac{x e^{-(n+2)x}}{n+2} \right]_0^\infty + \frac{1}{n+2} \int_0^\infty e^{-(n+2)x} dx \right) \\ &= \sum_{n=0}^\infty (-1)^n \left[ -\frac{e^{-(n+2)x}}{(n+2)^2} \right]_0^\infty \\ &= \sum_{n=0}^\infty \frac{(-1)^n}{(n+2)^2} = \sum_{n=2}^\infty (-1)^n \frac{1}{n^2} = 1 - \frac{\pi^2}{12} \end{aligned}$$

**Unjustified  $\lim / \int$  interchange**

## Level of Rigor Comments

$$\int_0^1 \frac{\ln(x)}{x+1} dx = -\frac{\pi^2}{12} \quad \text{Fully rigorous}$$

$$\int_0^\infty \frac{x}{e^x(e^x+1)} dx = 1 - \frac{\pi^2}{12} \quad \text{Unjustified } \lim / \int \text{ interchange}$$

$$\int_0^1 \frac{\tan^{-1}(x)}{x} dx = G \quad \text{Two unjustified } \lim / \int \text{ interchanges}$$

$$\int_0^\infty \frac{\ln(x)}{1+x^2} dx = G \quad \text{lim/} \int \text{ interchange assumption made}$$

$$\int_0^1 \frac{x \ln^3(x)}{1-x^4} dx = -\frac{\pi^4}{256} \quad [a, b] \text{ to } (0, \infty) \text{ ext'n assumption made}$$

$$\int_0^1 \frac{x \ln^5(x)}{x+1} dx = -\frac{31\pi^6}{352} \quad \text{lim/} \int \text{ interchange assumption made}$$

$$\int_0^1 \frac{\tan^{-1}(x)}{x} dx = G \quad \text{example in subject notes}$$

Example 6.13. From the integral

$$\tan^{-1} x = \int_0^x \frac{dy}{1+y^2}$$

and the expansion

$$\frac{1}{1+y^2} = 1 - y^2 + y^4 - y^6 + \dots, \quad |y| < 1,$$

we have the series

$$\begin{aligned} \tan^{-1} x &= \int_0^x (1 - y^2 + y^4 - y^6 + \dots) dy \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots, \quad |x| < 1. \end{aligned}$$

We have seen this before and the series converges uniformly within its radius of convergence. Thus

$$\begin{aligned} \int_0^1 \frac{\tan^{-1} x}{x} dx &= \int_0^1 \left( 1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \dots \right) dx \\ &= \left[ x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7^2} + \dots \right]_0^1 \\ &= \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^2}. \end{aligned}$$

**Two unjustified  $\lim / \int$  interchanges**

## *Position in the Notes*

*The end of these slides and the  
Week 9 class*

**CORRESPOND TO**

*the second-last page of the notes.*

*The Most Relevant Exercises  
(from the subject exercise sets)*

**K.11., L.4., L.5., L.6.**

## *Position in the Notes*

*The end of these slides and the  
Week 9 class*

**CORRESPOND TO**

*the second-last page of the notes.*

*However, solving integrals using Leibniz's rule  
still to be done NEXT WEEK.*

*If significant time remaining then...*

*possibly have a WHITEBOARD  
revision of*

***integration by  
substitution.***

