

35007

Solving Integrals Using Series

Professor Matt Wand

University of Technology Sydney

WANTED:

$$I = \int_0^{\infty} \frac{x^3}{e^x - 1} dx$$

Possibly useful first steps: $\int_0^\infty \frac{x^3}{e^x - 1} dx$

For all $x > 0$, the **geometric series** result

$$\frac{1}{1-r} = 1 + r + r^2 + \dots, \quad |r| < 1, \text{ gives}$$

$$\frac{1}{e^x - 1} = e^{-x} \left(\frac{1}{1 - e^{-x}} \right) = e^{-x} \sum_{k=0}^{\infty} (e^{-x})^k$$

$$= e^{-x} \lim_{n \rightarrow \infty} \sum_{k=0}^n (e^{-x})^k.$$

Continuing the quest to solve $\int_0^\infty \frac{x^3}{e^x - 1} dx$

From the previous slide

$$\frac{x^3}{e^x - 1} = x^3 e^{-x} \lim_{n \rightarrow \infty} \sum_{k=0}^n (e^{-x})^k, \quad x > 0.$$

Therefore

$$I = \int_0^\infty \lim_{n \rightarrow \infty} f_n(x) dx$$

where

$$f_n(x) = x^3 \sum_{k=0}^n e^{-(k+1)x}.$$

Proving Monotonicity of

$$f_n(x) = x^3 \sum_{k=0}^n e^{-(k+1)x}$$

For all $x > 0$ and $n \in \mathbb{N}$

$$\begin{aligned} f_{n+1}(x) - f_n(x) &= \sum_{k=0}^{n+1} e^{-(k+1)x} - \sum_{k=0}^n e^{-(k+1)x} \\ &= e^{-(n+2)x} > 0. \end{aligned}$$

Hence

$$f_{n+1}(x) > f_n(x).$$

***Tricky aspects
with respect to
limit / integral
interchange***

The Story So Far...

$$I = \int_0^{\infty} \lim_{n \rightarrow \infty} f_n(x) dx \quad \text{where}$$

$$f_{n+1}(x) > f_n(x) \quad \text{on} \quad (0, \infty).$$

The Story So Far...

$$I = \int_0^{\infty} \lim_{n \rightarrow \infty} f_n(x) dx \quad \text{where}$$

$$f_{n+1}(x) > f_n(x) \quad \text{on} \quad (0, \infty).$$

If, instead, we were dealing with

$$I = \int_a^b \lim_{n \rightarrow \infty} f_n(x) dx \quad \text{where}$$
$$f_{n+1}(x) > f_n(x) \quad \text{on} \quad [a, b]$$

then we could justify $I = \lim_{n \rightarrow \infty} \int_a^b f_n(x) dx.$

The Story So Far...

$$I = \int_0^{\infty} \lim_{n \rightarrow \infty} f_n(x) dx \quad \text{where}$$

$$f_{n+1}(x) > f_n(x) \quad \text{on} \quad (0, \infty).$$

If, instead, we were dealing with

$$I = \int_a^b \lim_{n \rightarrow \infty} f_n(x) dx \quad \text{where}$$
$$f_{n+1}(x) > f_n(x) \quad \text{on} \quad [a, b]$$

then we could justify $I = \lim_{n \rightarrow \infty} \int_a^b f_n(x) dx$.

For now we will proceed with $(0, \infty)$ treated like $[a, b]$ (a and b finite), and discuss justification later.

Next steps after limit / integral interchange
(assumed justifiable for now)

$$\begin{aligned} I &= \int_0^{\infty} \lim_{n \rightarrow \infty} f_n(x) \, dx \\ &= \lim_{n \rightarrow \infty} \int_0^{\infty} f_n(x) \, dx \\ &= \lim_{n \rightarrow \infty} \int_0^{\infty} x^3 \sum_{k=0}^n e^{-(k+1)x} \, dx \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^n \int_0^{\infty} x^3 e^{-(k+1)x} \, dx \\ &= \sum_{k=0}^{\infty} \int_0^{\infty} x^3 e^{-(k+1)x} \, dx \end{aligned}$$

The Story So Far...

$$I = \sum_{k=0}^{\infty} \int_0^{\infty} x^3 e^{-(k+1)x} dx$$

The Story So Far...

$$I = \sum_{k=0}^{\infty} \int_0^{\infty} x^3 e^{-(k+1)x} dx$$

Substitution time:

$$u = (k + 1)x \implies x = \frac{u}{k + 1} \implies dx = \frac{du}{k + 1}.$$

$$x = 0 \implies u = 0 \quad \text{and} \quad x = \infty \implies u = \infty.$$

The Story So Far...

$$I = \sum_{k=0}^{\infty} \int_0^{\infty} x^3 e^{-(k+1)x} dx$$

Substitution time:

$$u = (k+1)x \implies x = \frac{u}{k+1} \implies dx = \frac{du}{k+1}.$$

$$x = 0 \implies u = 0 \quad \text{and} \quad x = \infty \implies u = \infty.$$

$$\rightarrow I = \sum_{k=0}^{\infty} \frac{1}{(k+1)^4} \int_0^{\infty} u^3 e^{-u} du = 3! \sum_{k=0}^{\infty} \frac{1}{(k+1)^4}$$

Final Steps...

Letting $j = k - 1$:

$$I = 6 \sum_{k=0}^{\infty} \frac{1}{(k+1)^4} = 6 \sum_{j=1}^{\infty} \frac{1}{j^4}.$$

Final Steps...

Letting $j = k - 1$:

$$I = 6 \sum_{k=0}^{\infty} \frac{1}{(k+1)^4} = 6 \sum_{j=1}^{\infty} \frac{1}{j^4}.$$

From **equation (1.2) on page 18** of the notes

$$\sum_{j=1}^{\infty} \frac{1}{j^4} = \frac{\pi^4}{90}.$$

Final Steps...

Letting $j = k - 1$:

$$I = 6 \sum_{k=0}^{\infty} \frac{1}{(k+1)^4} = 6 \sum_{j=1}^{\infty} \frac{1}{j^4}.$$

From **equation (1.2) on page 18** of the notes

$$\sum_{j=1}^{\infty} \frac{1}{j^4} = \frac{\pi^4}{90}.$$

$$\longrightarrow I = \int_0^1 \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}.$$

*However, we
still have not justified
the limit / integral
interchange due to
 $(0, \infty)$ versus $[a, b]$.*

Using the Weierstrass M-test to get uniform convergence over $(0, \infty)$.

$$f_n(x) = \sum_{k=0}^n x^3 e^{-(k+1)x}, \quad x > 0.$$

Several lines of analysis (omitted here) leads to

$$\left| x^3 e^{-(k+1)x} \right| \leq \frac{27}{e^3 (k+1)^3} \quad \text{for all } x > 0 \text{ and } k \in \mathbb{N}.$$

Since

$$\sum_{k=0}^{\infty} \frac{27}{e^3 (k+1)^3} = \frac{27}{e^3} \sum_{k=1}^{\infty} \frac{1}{k^3} \quad \text{converges,}$$

by the Weierstrass M-test f_n converges uniformly to

$$f(x) = \frac{x^3}{e^x - 1} \quad \text{over } (0, \infty).$$

We now have

f_n converges uniformly to f over $(0, \infty)$.

and want to **FULLY JUSTIFY**

$$\lim_{n \rightarrow \infty} \int_0^{\infty} f_n(x) dx = \int_0^{\infty} \lim_{n \rightarrow \infty} f_n(x) dx.$$

We now have

f_n converges uniformly to f over $(0, \infty)$.

and want to **FULLY JUSTIFY**

$$\lim_{n \rightarrow \infty} \int_0^{\infty} f_n(x) dx = \int_0^{\infty} \lim_{n \rightarrow \infty} f_n(x) dx.$$

However **Theorem 5.10** with $[a, b]$ a **bounded** interval does not cover the $(0, \infty)$ case. **Uniform convergence over $(0, \infty)$ is not enough** and **advanced literature** says we **also** need to prove a condition such as

$$\lim_{b \rightarrow \infty} \sup \left\{ \int_{1/b}^b |f_n(x)| dx : n \in \mathbb{N} \right\} = 0.$$

Notes and Exercise Examples

$$\int_0^1 \frac{\ln(x)}{x+1} dx = -\frac{\pi^2}{12}$$

pages 91–92

$$\int_0^\infty \frac{x}{e^x(e^x+1)} dx = 1 - \frac{\pi^2}{12}$$

page 92

$$\int_0^1 \frac{\tan^{-1}(x)}{x} dx = G \quad [\text{Catalan's constant}]$$

page 93

$$\int_1^\infty \frac{\ln(x)}{1+x^2} dx = G$$

Exercise L.4.

$$\int_0^1 \frac{x \ln^3(x)}{1-x^4} dx = -\frac{\pi^4}{256}$$

Exercise L.5.

$$\int_0^1 \frac{x \ln^5(x)}{x+1} dx = -\frac{31\pi^6}{352}$$

Exercise L.6.

Level of Rigor Comments

$$\int_0^1 \frac{\ln(x)}{x+1} dx = -\frac{\pi^2}{12}$$

Fully rigorous

$$\int_0^\infty \frac{x}{e^x(e^x+1)} dx = 1 - \frac{\pi^2}{12}$$

Unjustified $\lim/\int$ interchange

$$\int_0^1 \frac{\tan^{-1}(x)}{x} dx = G$$

Two unjustified $\lim/\int$ interchanges

$$\int_1^\infty \frac{\ln(x)}{1+x^2} dx = G$$

$\lim/\int$ interchange assumption made

$$\int_0^1 \frac{x \ln^3(x)}{1-x^4} dx = -\frac{\pi^4}{256}$$

$[a, b]$ to $(0, \infty)$ ext'n assumption made

$$\int_0^1 \frac{x \ln^5(x)}{x+1} dx = -\frac{31\pi^6}{352}$$

$\lim/\int$ interchange assumption made

$$\int_0^{\infty} \frac{x}{e^x(e^x+1)} dx = 1 - \frac{\pi^2}{12} \text{ example in subject notes}$$

Example 6.12. Now let us use the same method to evaluate

$$I = \int_0^{\infty} \frac{xe^{-x}}{e^x + 1} dx. \quad (6.36)$$

To use a geometric series we write

$$\frac{1}{e^x + 1} = \frac{1}{e^x(1 + e^{-x})} = e^{-x} (1 - e^{-x} + e^{-2x} - e^{-3x} + \dots). \quad (6.37)$$

This converges because $|e^{-x}| < 1$ for $x > 0$. Hence

$$\begin{aligned} I &= \int_0^{\infty} xe^{-2x} \sum_{n=0}^{\infty} (-1)^n e^{-nx} dx \\ &= \sum_{n=0}^{\infty} (-1)^n \int_0^{\infty} xe^{-(n+2)x} dx \\ &= \sum_{n=0}^{\infty} (-1)^n \left(\left[\frac{xe^{-(n+2)x}}{n+2} \right]_0^{\infty} + \frac{1}{n+2} \int_0^{\infty} e^{-(n+2)x} dx \right) \\ &= \sum_{n=0}^{\infty} (-1)^n \left[-\frac{e^{-(n+2)x}}{(n+2)^2} \right]_0^{\infty} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)^2} = \sum_{n=2}^{\infty} (-1)^n \frac{1}{n^2} = 1 - \frac{\pi^2}{12}. \end{aligned}$$

Unjustified $\lim / \int$ interchange

$$\int_0^1 \frac{\tan^{-1}(x)}{x} dx = G \quad \text{example in subject notes}$$

Example 6.13. From the integral

$$\tan^{-1} x = \int_0^x \frac{dy}{1+y^2}$$

and the expansion

$$\frac{1}{1+y^2} = 1 - y^2 + y^4 - y^6 + \cdots, \quad |y| < 1,$$

we have the series

$$\begin{aligned} \tan^{-1} x &= \int_0^x (1 - y^2 + y^4 - y^6 + \cdots) dy \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{y^7}{7} + \cdots, \quad |x| < 1. \end{aligned}$$

We have seen this before and the series converges uniformly within its radius of convergence. Thus

$$\begin{aligned} \int_0^1 \frac{\tan^{-1} x}{x} dx &= \int_0^1 \left(1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{y^6}{7} + \cdots \right) dx \\ &= \left[x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7} + \cdots \right]_0^1 \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}. \end{aligned}$$

Two unjustified $\lim / \int$ interchanges

Position in the Notes

***The end of these slides and the
Week 9 class***

CORRESPOND TO

the second-last page of the notes.

Position in the Notes

***The end of these slides and the
Week 9 class***

CORRESPOND TO

the second-last page of the notes.

***However, solving integrals using Leibniz's rule
still to be done NEXT WEEK.***

The Most Relevant Exercises
(from the subject exercise sets)

K.11., L.4., L.5., L.6.

If significant time remaining then...

***possibly have a WHITEBOARD
revision of***

***integration by
substitution.***

