

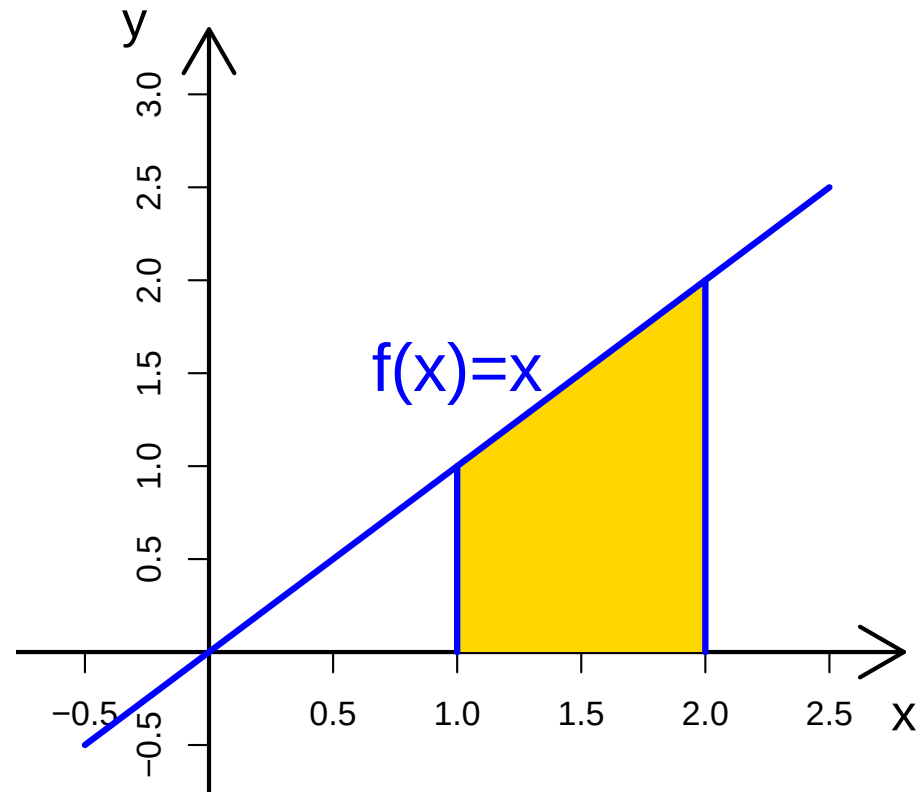
35007

Integration
and
Real Analysis

Professor Matt Wand

University of Technology Sydney

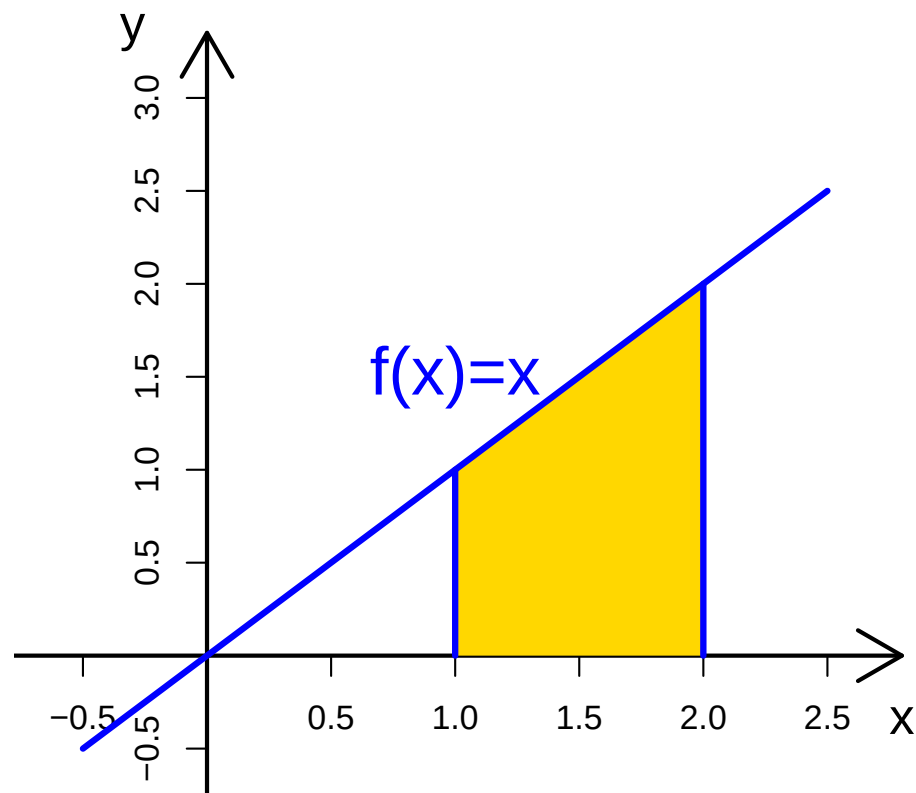
High School / Maths 1 Level



$$\int_1^2 x \, dx = \left[\frac{x^2}{2} \right]_1^2 = \frac{3}{2} = \text{area of shaded region}$$

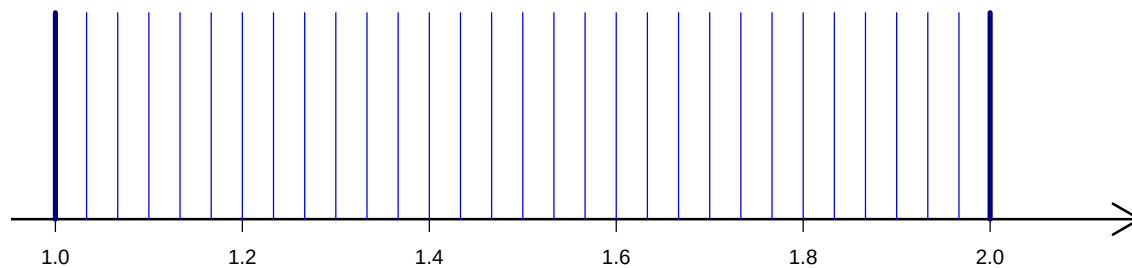
Real Analysis Level

*What is the formal
Riemann integral
of $f(x) = x$ over $[1, 2]$?*

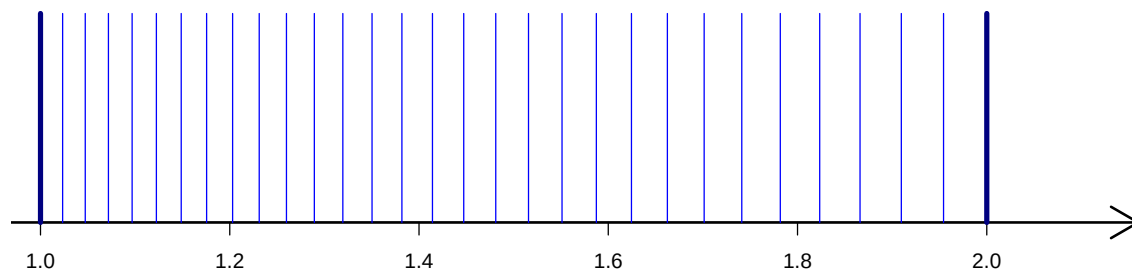


Partitions of $[1,2]$

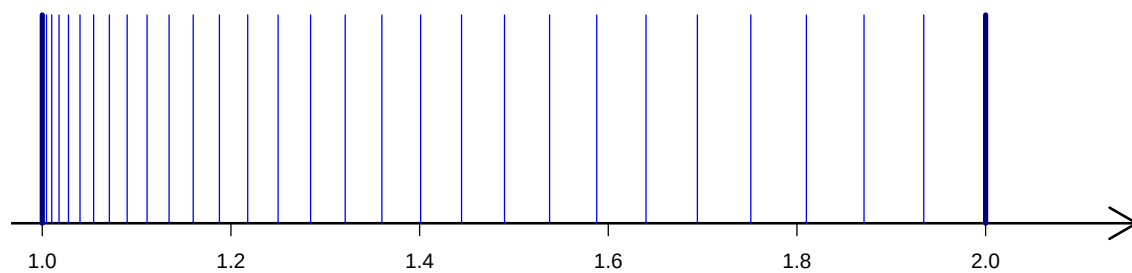
Arithmetic Partition



Geometric Partition



Quadratic Partition



The Three Partitions in Algebraic Form, $n \in \mathbb{N}$

Arithmetic Partition

$$\mathcal{P}_A = \left\{ 1, 1 + \frac{1}{n}, 1 + \frac{2}{n}, \dots, 1 + \frac{n-1}{n}, 2 \right\}$$

Geometric Partition

$$\mathcal{P}_G = \left\{ 1, 2^{1/n}, 2^{2/n}, \dots, 2^{(n-1)/n}, 2 \right\}$$

Quadratic Partition

$$\mathcal{P}_Q = \left\{ 1, 1 + \frac{1^2}{n^2}, 1 + \frac{2^2}{n^2}, \dots, 1 + \frac{(n-1)^2}{n^2}, 2 \right\}$$

WANTED: The Riemann integral of

$$f(x) = x \text{ **over** } [1, 2].$$

WANTED: The Riemann integral of

$$f(x) = x \text{ **over** } [1, 2].$$

***First we need to choose a
partition of $[1, 2]$ such as***

$$\mathcal{P}_A, \mathcal{P}_G, \mathcal{P}_Q$$

***or one of (infinitely) many other
partitions.***

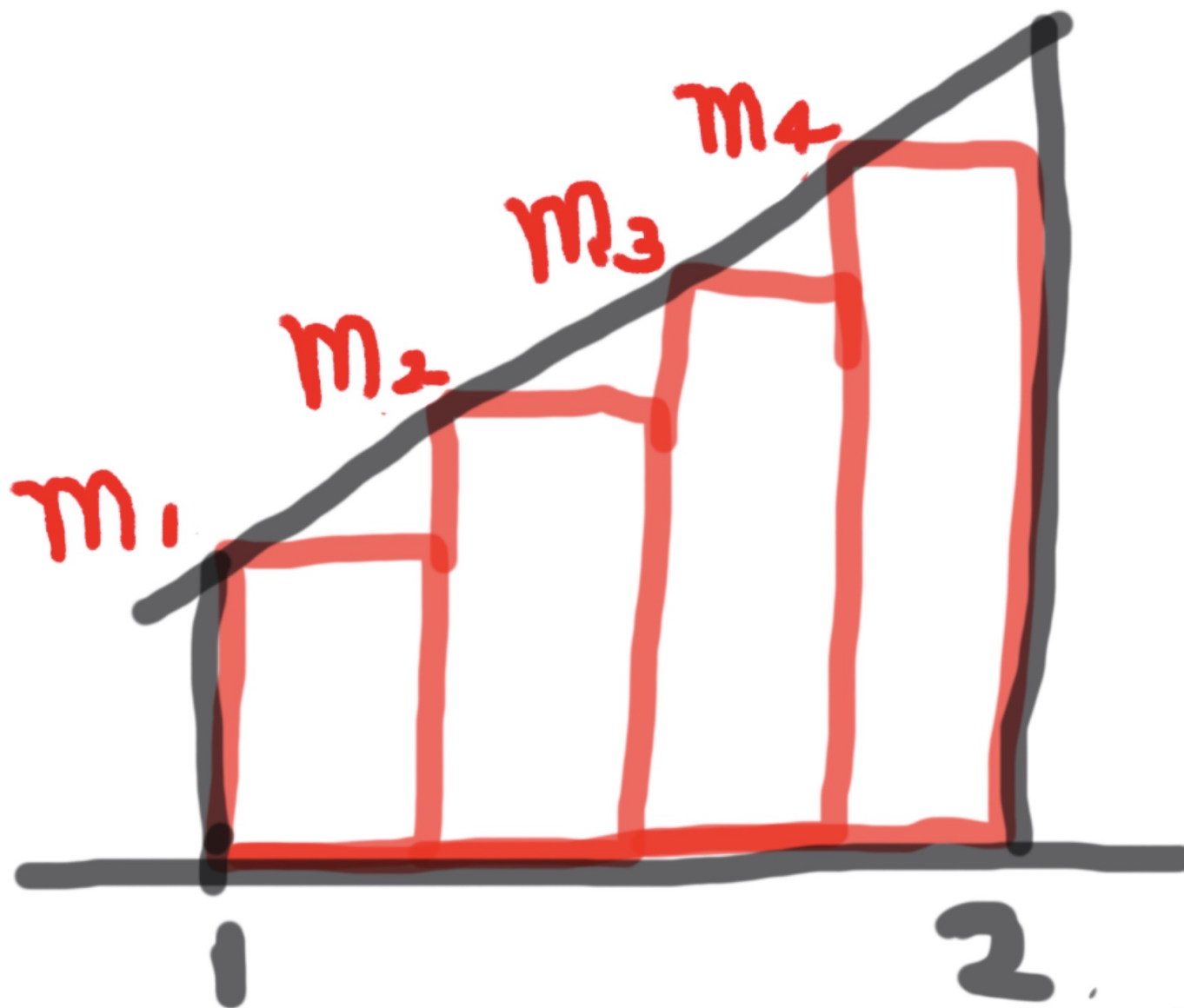
NEXT UP...

n_i *and* M_i

The m_i

Quantities

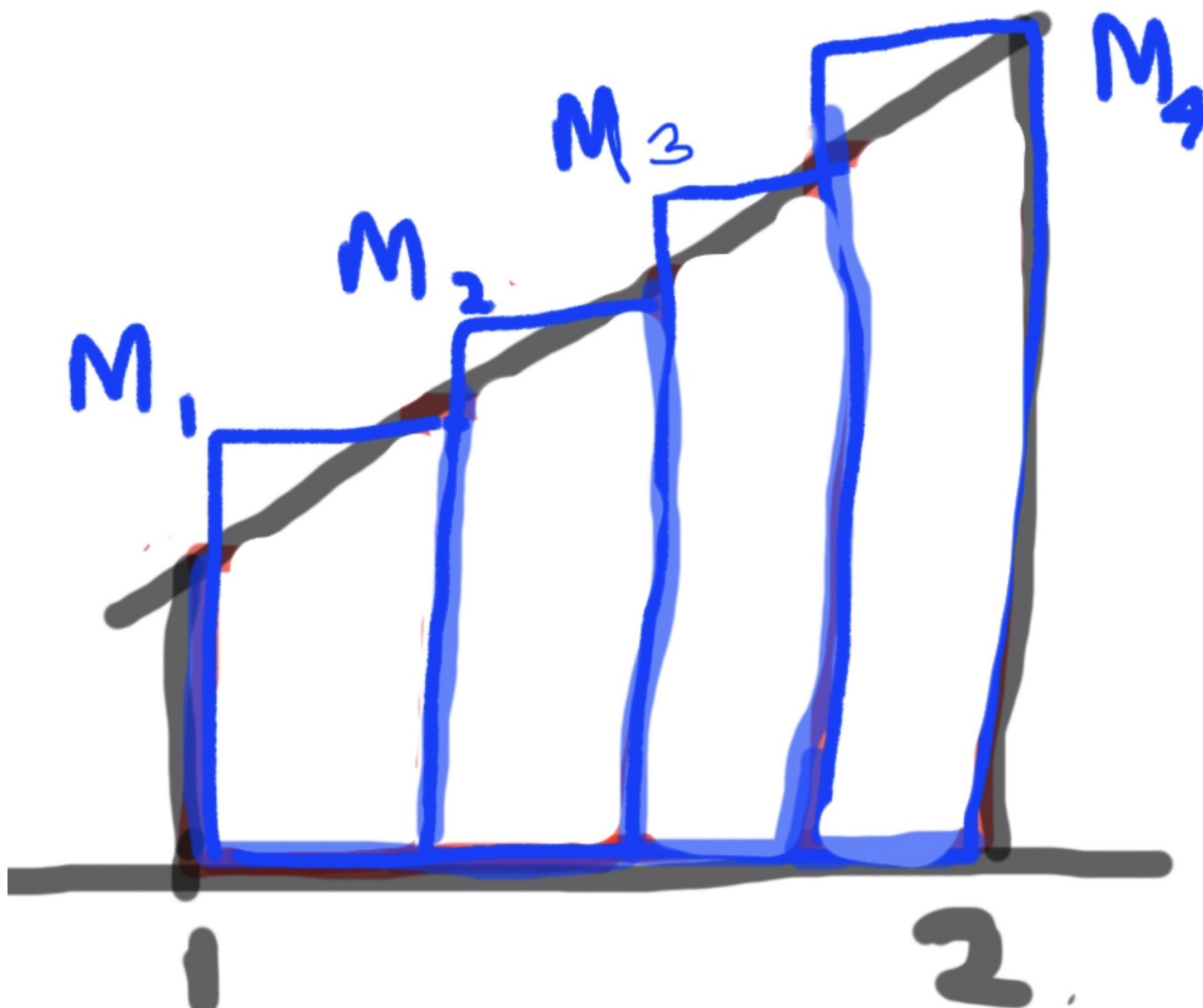
The m_i Quantities for $f(x) = x$ on $[1, 2]$



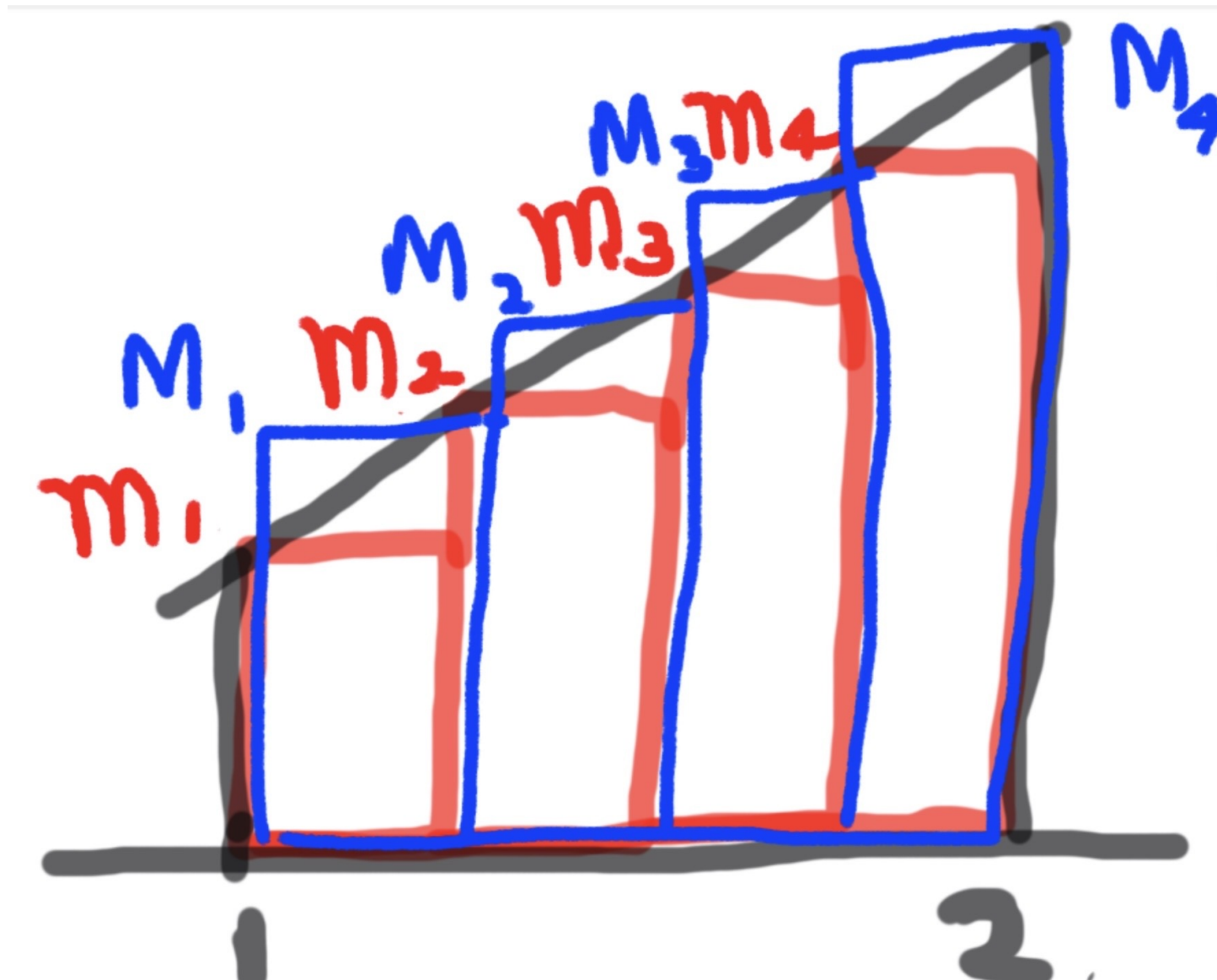
The M_i

Quantities

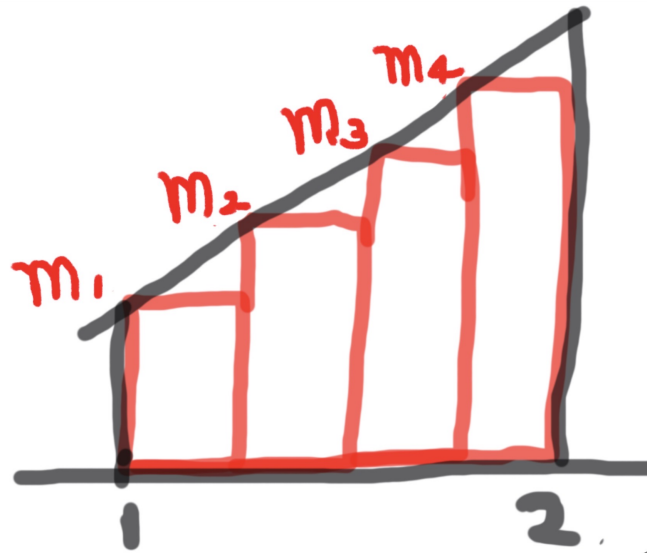
The M_i Quantities for $f(x) = x$ on $[1, 2]$



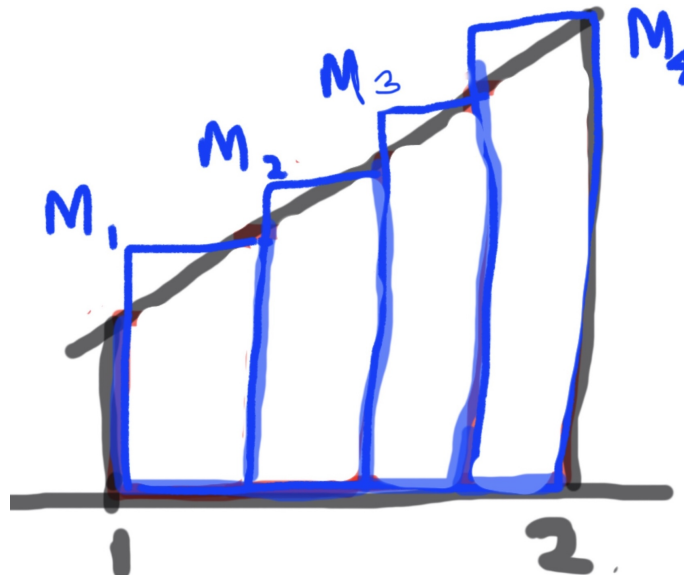
The m_i and M_i Quantities Together



The $L(f, \mathcal{P})$
and $U(f, \mathcal{P})$
Quantities



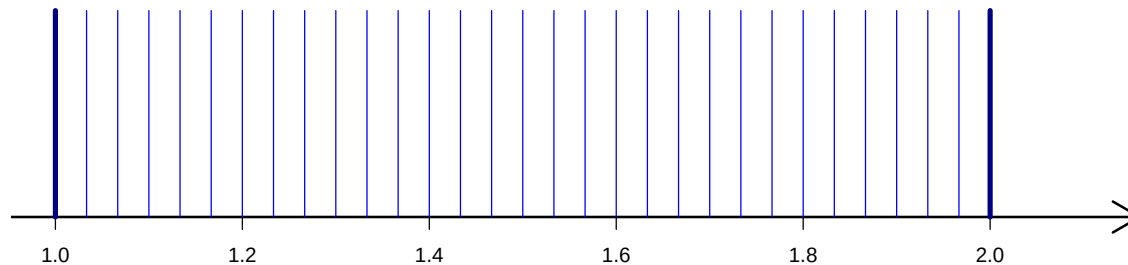
$L(f, \mathcal{P})$ = total area of RED rectangles



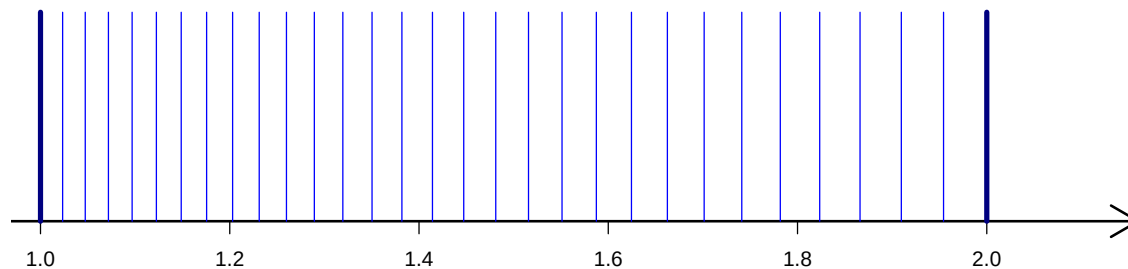
$U(f, \mathcal{P})$ = total area of BLUE rectangles

Partition-Dependent Answers

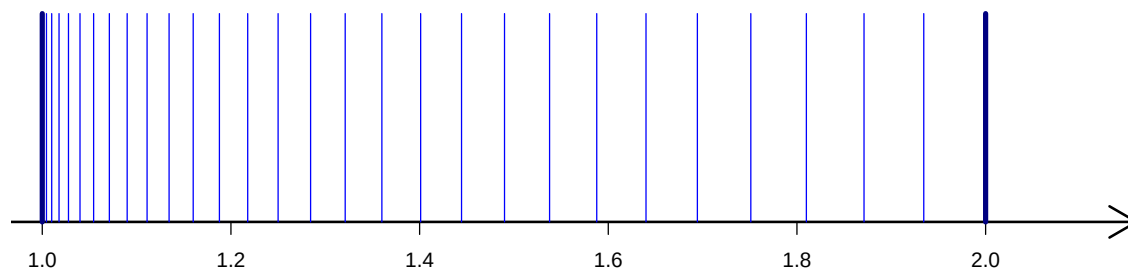
Arithmetic Partition



Geometric Partition



Quadratic Partition



Partition-Dependent Answers with n Rectangles

$$L(f, \mathcal{P}_A) = \frac{3}{2} - \frac{1}{2n}$$

$$U(f, \mathcal{P}_A) = \frac{3}{2} + \frac{1}{2n}$$

$$L(f, \mathcal{P}_G) = \frac{3}{1 + 2^{1/n}}$$

$$U(f, \mathcal{P}_G) = \frac{3}{1 + 2^{-1/n}}$$

$$L(f, \mathcal{P}_Q) = \frac{3}{2} - \frac{2}{3n} + \frac{1}{6n^3}$$

$$U(f, \mathcal{P}_Q) = \frac{3}{2} + \frac{2}{3n} - \frac{1}{6n^3}$$

The $\int_1^2 f$

and $\overline{\int_1^2 f}$

Quantities

Arithmetic Partition

$$\begin{aligned}\left(\underline{\int_1^2} f\right)_A &= \sup\{L(f, \mathcal{P}_A) : n \in \mathbb{N}\} \\ &= \sup\left\{\frac{3}{2} - \frac{1}{2n} : n \in \mathbb{N}\right\} = \frac{3}{2}.\end{aligned}$$

$$\begin{aligned}\left(\overline{\int_1^2} f\right)_A &= \inf\{U(f, \mathcal{P}_A) : n \in \mathbb{N}\} \\ &= \inf\left\{\frac{3}{2} + \frac{1}{2n} : n \in \mathbb{N}\right\} = \frac{3}{2}.\end{aligned}$$

Geometric Partition

$$\begin{aligned}\left(\int_1^2 f\right)_G &= \sup\{L(f, \mathcal{P}_G) : n \in \mathbb{N}\} \\ &= \sup\left\{\frac{3}{1 + 2^{1/n}} : n \in \mathbb{N}\right\} = \frac{3}{2}.\end{aligned}$$

$$\begin{aligned}\left(\overline{\int_1^2 f}\right)_G &= \inf\{U(f, \mathcal{P}_G) : n \in \mathbb{N}\} \\ &= \inf\left\{\frac{3}{1 + 2^{-1/n}} : n \in \mathbb{N}\right\} = \frac{3}{2}.\end{aligned}$$

Quadratic Partition

$$\begin{aligned}\left(\underline{\int_1^2} f\right)_Q &= \sup\{L(f, \mathcal{P}_Q) : n \in \mathbb{N}\} \\ &= \sup\left\{\frac{3}{2} - \frac{2}{3n} + \frac{1}{6n^3} : n \in \mathbb{N}\right\} = \frac{3}{2}.\end{aligned}$$

$$\begin{aligned}\left(\overline{\int_1^2} f\right)_Q &= \inf\{U(f, \mathcal{P}_Q) : n \in \mathbb{N}\} \\ &= \inf\left\{\frac{3}{2} + \frac{2}{3n} - \frac{1}{6n^3} : n \in \mathbb{N}\right\} = \frac{3}{2}.\end{aligned}$$

***Reduction From
(Infinitely) Many
Partition Families
to One***

Use of Riemann's Criterion

The Arithmetic Partition Case

$$U(f, \mathcal{P}_A) - L(f, \mathcal{P}_A) = \frac{3}{2} + \frac{2}{n} - \left(\frac{3}{2} - \frac{2}{n} \right) = \frac{4}{n}.$$

Use of Riemann's Criterion

The Arithmetic Partition Case

$$U(f, \mathcal{P}_A) - L(f, \mathcal{P}_A) = \frac{3}{2} + \frac{2}{n} - \left(\frac{3}{2} + \frac{2}{n} \right) = \frac{4}{n}.$$

Hence, for any $\varepsilon > 0$,

$$n > \frac{4}{\varepsilon} \quad \implies \quad U(f, \mathcal{P}_A) - L(f, \mathcal{P}_A) < \varepsilon.$$

Use of Riemann's Criterion

The Arithmetic Partition Case

$$U(f, \mathcal{P}_A) - L(f, \mathcal{P}_A) = \frac{3}{2} + \frac{2}{n} - \left(\frac{3}{2} - \frac{2}{n} \right) = \frac{4}{n}.$$

Hence, for any $\varepsilon > 0$,

$$n > \frac{4}{\varepsilon} \quad \implies \quad U(f, \mathcal{P}_A) - L(f, \mathcal{P}_A) < \varepsilon.$$

Riemann's Criterion (**Theorem 4.2**) implies that it is sufficient to use only **arithmetic partitions**.

Formal Riemann Integration Steps for General $\int_a^b f(x) dx$

1. *Consider some partition families such as **arithmetic** or **geometric** and hope that one of them gives a nice **Riemann's Criterion** outcome. **This may involve some trial and error.***
2. ***Establish Riemann's Criterion for one partition family. Use this partition family, $\mathcal{P}$, from now on.***

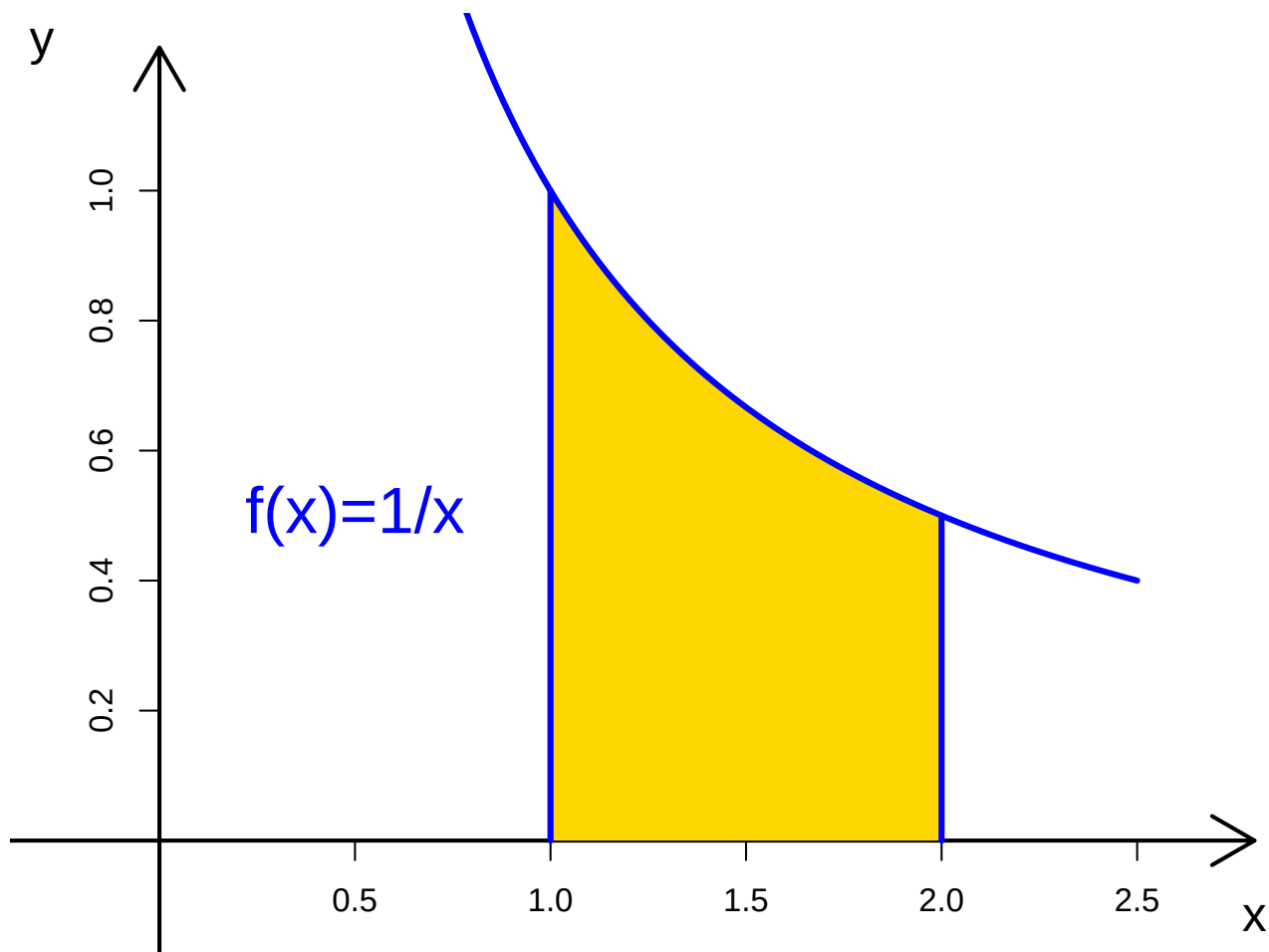
3. ***Toss a coin.***

If HEADS then obtain $L(f, \mathcal{P})$ and then obtain $\underline{\int_a^b} f$.

If TAILS then obtain $U(f, \mathcal{P})$ and then obtain $\overline{\int_a^b} f$.

4. *$\int_a^b f(x) dx$ **equals the answer from Step 3.***

One More Example



How to formally obtain $\int_1^2 \frac{1}{x} dx$.

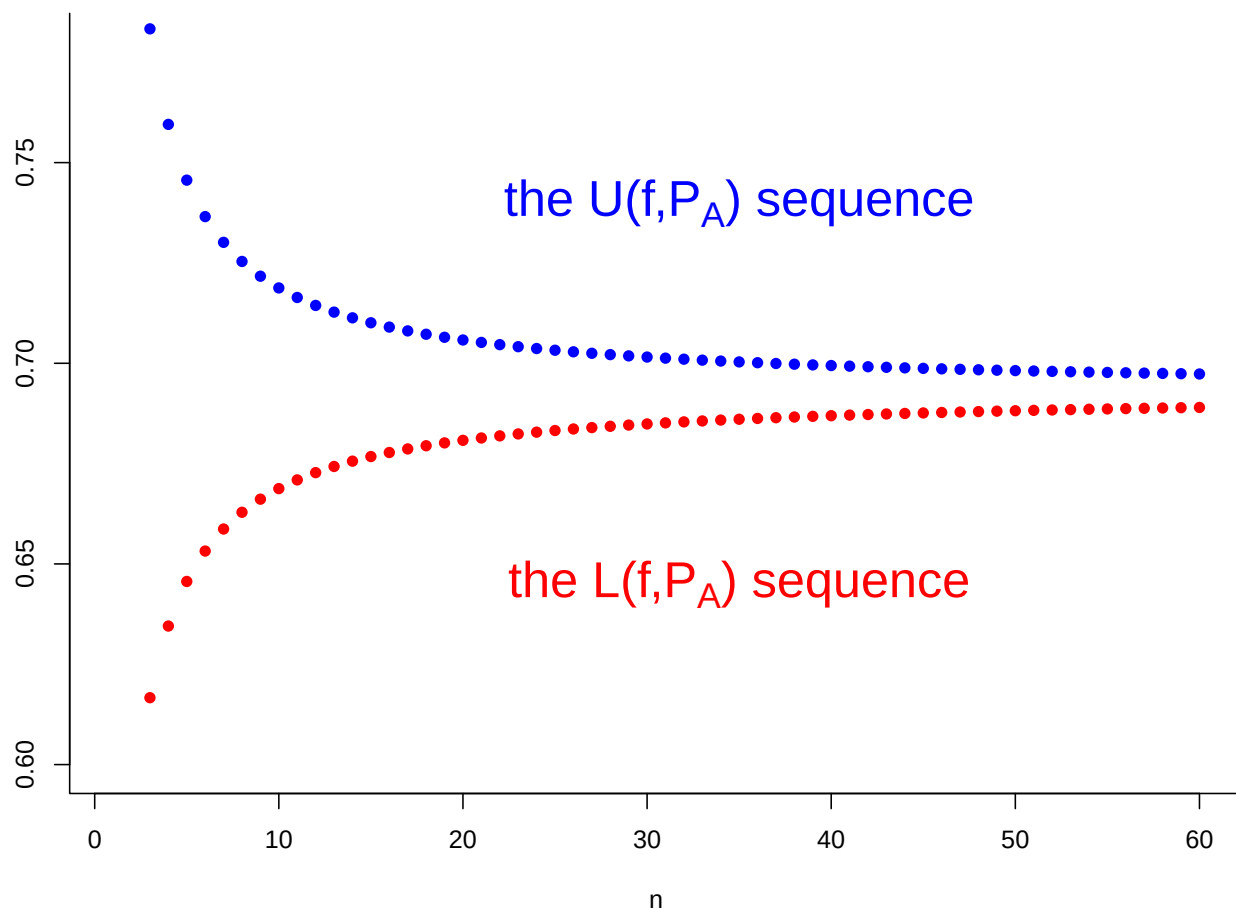
Arithmetic Partition Results

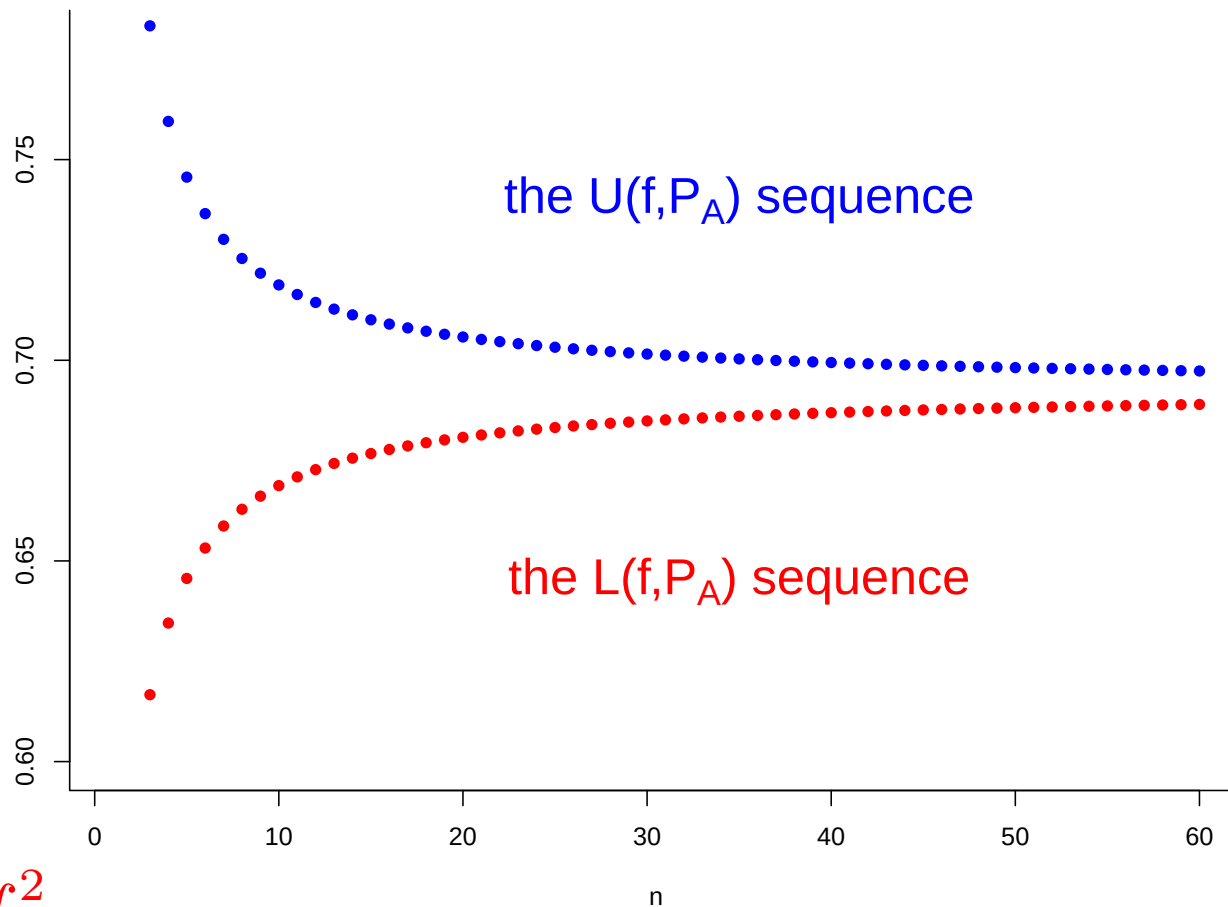
$$U(f, \mathcal{P}_A) - L(f, \mathcal{P}_A) = \frac{1}{2n}$$

$\Rightarrow$ **Riemann's Criterion holds with** $n > \frac{1}{2\varepsilon}$.

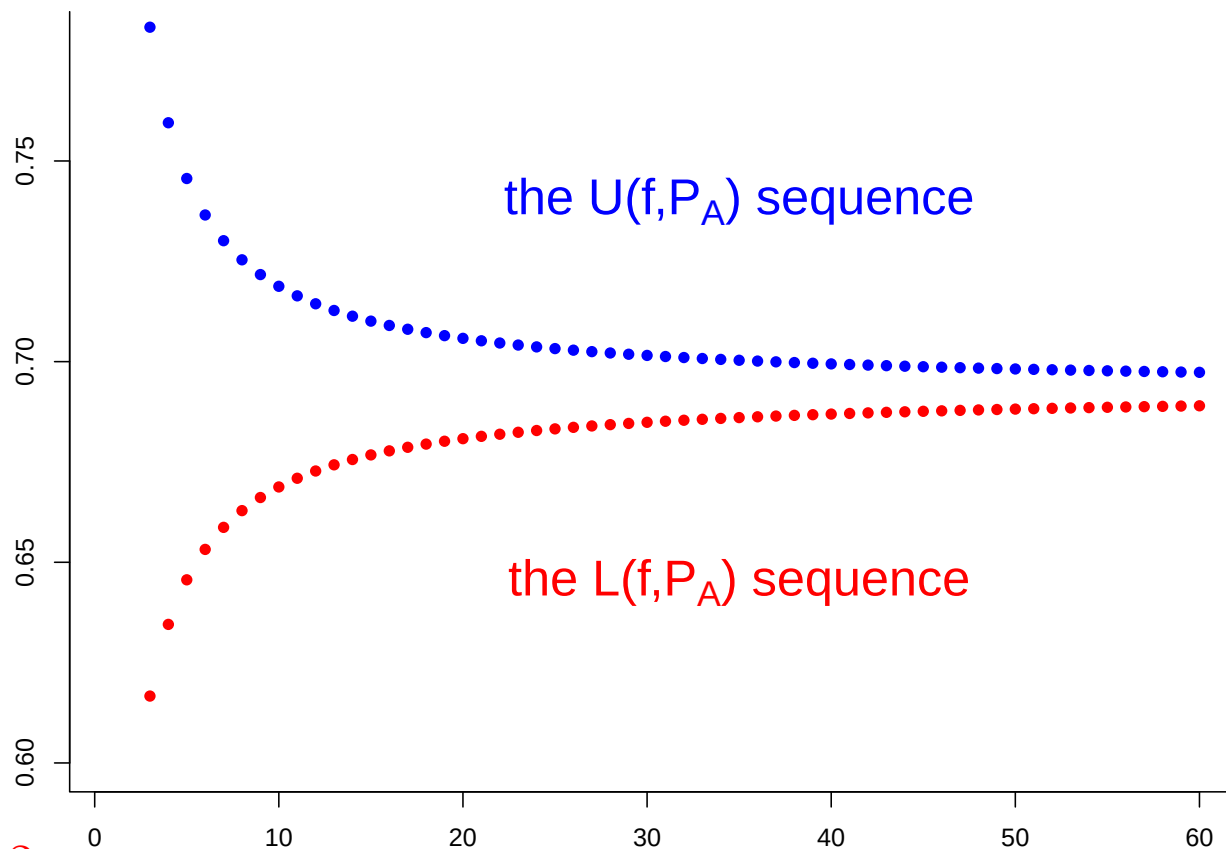
Proceed now with the arithmetic partition







$$\begin{aligned}
 \int_1^2 f &= \sup \{ L(f, \mathcal{P}_A) : n \in \mathbb{N} \} \\
 &= \sup \left\{ \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} : n \in \mathbb{N} \right\} \\
 &= 0.6931471805599452862 \dots
 \end{aligned}$$



$$\begin{aligned}
 \int_1^2 f &= \sup \{ L(f, \mathcal{P}_A) : n \in \mathbb{N} \} \\
 &= \sup \left\{ \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} : n \in \mathbb{N} \right\} \\
 &= 0.6931471805599452862 \dots
 \end{aligned}$$

$$= \ln(2) \quad \textbf{essentially BY DEFINITION}$$

OR, IF YOU PREFER:

$$\begin{aligned}\overline{\int_1^2 f} &= \inf \{U(f, \mathcal{P}_A) : n \in \mathbb{N}\} \\ &= \inf \left\{ \frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{2n-1} : n \in \mathbb{N} \right\} \\ &= 0.6931471805599452862 \dots = \ln(2)\end{aligned}$$

OR, IF YOU PREFER:

$$\begin{aligned}\overline{\int_1^2} f &= \inf \{ U(f, \mathcal{P}_A) : n \in \mathbb{N} \} \\ &= \inf \left\{ \frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{2n-1} : n \in \mathbb{N} \right\} \\ &= 0.6931471805599452862 \dots = \ln(2)\end{aligned}$$

THUS:

$$\int_1^2 \frac{1}{x} dx = \underline{\int_1^2} f = \overline{\int_1^2} f = \ln(2).$$

The Formal Real Analysis
Definition of 2^3

REVISITED...

The Formal Real Analysis

Definition of 2^3

$$\begin{aligned} 2^3 &= \exp \left(3 \ln(2) \right) \\ &= \ln^{-1} \left(3 \ln(2) \right) \\ &= \left[x > 0 \text{ such that } \int_1^x \frac{1}{t} dt = 3 \int_1^2 \frac{1}{t} dt \right]. \end{aligned}$$

The Formal Real Analysis

Definition of 2^3

$$2^3 = \exp\left(3 \ln(2)\right)$$

$$= \ln^{-1}\left(3 \ln(2)\right)$$

$$= \left[x > 0 \text{ such that } \int_1^x \frac{1}{t} dt = 3 \int_1^2 \frac{1}{t} dt \right]$$

$$= \left[x > 0 \text{ such that } \sup \left\{ \sum_{k=1}^n \frac{x-1}{n+k(x-1)} : n \in \mathbb{N} \right\} \right]$$

$$= 3 \sup \left\{ \sum_{k=1}^n \frac{1}{n+k} : n \in \mathbb{N} \right\} \Big].$$

The Most Relevant Exercises
(from the current subject exercise sets)

Exercise Set J

*Some
Admin.*

Next Week's Assessments

Deadline Two of Assignment	Start of Class 8
Quiz 2 (★)	First Half of Class 8

(★) ***See the web-page for Quiz 2 information,
Practice Quiz 2 and solutions.***

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*⇒ 60% of assessment
achieved by this time next week!*

