

35007

Solving Integrals Using Leibniz's Rule

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Leibniz's Rule is . . .

1. *Concerned with conditions that allow integral / partial derivative interchanges of the form*

$$\frac{\partial}{\partial t} \int_a^b f(x, t) dx = \int_a^b \frac{\partial}{\partial t} f(x, t) dx.$$

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2. *Is the basis of a technique for solving particular types of integrals.*

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2. *Is the basis of a technique for solving particular types of integrals.*
3. *This approach is sometimes (in physics) called Feynman's Technique after 1965 physics Nobel Prize winner Richard Feynman (1918-1988) even though it has been used in maths since the 1700s.*

Illustrative Example

$$I = \int_0^1 \frac{x^\pi - 1}{\ln(x)} dx$$

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Leibniz's Rule *trick:* *write*

$$I = F(\pi) \quad \text{where} \quad F(t) = \int_0^1 \frac{x^t - 1}{\ln(x)} dx$$

Essence of the Leibniz's Rule Approach

1. *Use Leib. Rule to get a formula for $F'(t)$.*
2. *Use integration with respect to t to get a formula for $F(t)$.*
3. *$I = F(\pi)$.*

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$$F(t) = \int_0^1 \frac{x^t - 1}{\ln(x)} dx.$$

$$\Rightarrow F'(t) = \frac{\partial}{\partial t} \int_0^1 \frac{x^t - 1}{\ln(x)} dx = \int_0^1 \frac{\partial}{\partial t} \frac{x^t - 1}{\ln(x)} dx = \int_0^1 x^t dx = \frac{1}{t+1}.$$

$$\Rightarrow F(t) = \int \frac{1}{t+1} dt = \ln(t+1) \text{ [using the } F(0) = 0 \text{ boundary condition].}$$

$$\Rightarrow I = F(\pi) = \ln(\pi + 1).$$

***Slow
Motion
Replay***

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$$F(t) = \ln(t+1) \implies I = F(\pi) = \ln(\pi+1).$$

Result from Last Several Slides
Thanks to Leibniz's Rule

$$\int_0^1 \frac{x^\pi - 1}{\ln(x)} dx = \ln(\pi + 1)$$

A Second Illustrative Example

$$J = \int_0^{\infty} \frac{\sin(x)}{xe^{17x}} dx$$

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$$F(t) = \int_0^{\infty} \frac{\sin(x)}{xe^{tx}} dx$$

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[Fuller details of these steps on next slides.]

$$\Rightarrow F(t) = -\int \frac{1}{1+t^2} dt = \frac{\pi}{2} - \tan^{-1}(t) \text{ [using the } F(0) = \frac{\pi}{2}, \text{ shown soon]}$$

$$\Rightarrow J = F(17) = \frac{\pi}{2} - \tan^{-1}(17).$$

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$$\implies F'(t) = \int_0^{\infty} \frac{\sin(x)}{x} e^{-tx} (-x) dx = \int_0^{\infty} \{-\sin(x)\} e^{-tx} dx$$

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Integration by Parts Evaluation of $\int_0^{\infty} \{-\sin(x)\} e^{-tx} dx$

$$\int_0^{\infty} \{-\sin(x)\} e^{-tx} dx = \left[\cos(x) e^{-tx} \right]_0^{\infty} - \int_0^{\infty} \cos(x) e^{-tx} (-t) dx$$

$$= -1 + t \int_0^{\infty} \cos(x) e^{-tx} dx = -1 + t \left(\left[\sin(x) e^{-tx} \right]_0^{\infty} - \int_0^{\infty} \sin(x) e^{-tx} (-t) dx \right)$$

$$= -1 - t^2 \int_0^{\infty} \{-\sin(x)\} e^{-tx} dx \implies \int_0^{\infty} \{-\sin(x)\} e^{-tx} dx = -\frac{1}{1+t^2}$$

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$$F(0) = \int_0^{\infty} \frac{\sin(x)}{x} dx = \frac{\pi}{2} \text{ [Shown next slide.] } \implies F(t) = \frac{\pi}{2} - \tan^{-1}(t)$$

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$$\implies J = F(17) = \frac{\pi}{2} - \tan^{-1}(17).$$

Derivation of $\int_0^{\infty} \frac{\sin(x)}{x} dx = \frac{\pi}{2}$

$$\int_0^{\infty} e^{-xt} dt = \left[-\frac{1}{t} e^{-xt} \right]_0^{\infty} = \frac{1}{x}$$

$$\Rightarrow \frac{\sin(x)}{x} = \sin(x) \int_0^{\infty} e^{-xt} dt = \int_0^{\infty} \sin(x) e^{-xt} dt$$

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[But in the previous line we are assuming that use of Fubini's Theorem is justified.]

Using the earlier **integration by parts result** $\int_0^{\infty} \sin(x) e^{-xt} dx = \frac{1}{1+t^2}$.

Hence

$$\int_0^{\infty} \frac{\sin(x)}{x} dx = \int_0^{\infty} \frac{1}{1+t^2} dt = [\tan^{-1}(t)]_0^{\infty} = \frac{\pi}{2}.$$

Result from Last Several Slides
Thanks to Leibniz's Rule

$$\int_0^{\infty} \frac{\sin(x)}{xe^{17x}} dx = \frac{\pi}{2} - \tan^{-1}(17)$$

Some Closing Remarks (1/3)

1. *Theorem 6.1 of the subject notes gives sufficient conditions for **integral/ partial derivative** interchanging. An example is that*

$$\frac{\partial f(x, t)}{\partial t}$$

exists and is continuous over the integration region.

Some Closing Remarks (2/3)

2. Theorem 6.1 on Leibniz's Rule also covers the integral limits depending on t , such as

$$\frac{\partial}{\partial t} \int_{\frac{1}{2}\exp(-t)}^{\cosh(t)} \ln \left\{ \sin^4(t^3 + \sqrt{x(t+1)}) \right\} dx.$$

Some Closing Remarks (3/3)

*3. These slides, the subject notes and the exercise sets have **six examples** in total of using **Leibniz's Rule** to **solve integrals**.*

Finding additional examples is proving to be challenging.

At the time that these slides were finalised in Spring 2026, it is not clear to the lecturer that the principle has wide applicability.

Position in the Notes

***The end of these slides and the
Week 10 (pre-computer laboratory) class***

CORRESPOND TO

the END OF THE NOTES.

The Most Relevant Exercises
(from the subject exercise sets)

L.7., L.8., L.9., L.10.

*Some
Admin.*

1. *Subject Evaluation*

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2. *Final Examination*

1. *Subject Evaluation*

2. *Final Examination*

3. *Week 11 Class*

1. *Subject Evaluation*

2. *Final Examination*

3. *Week 11 Class*

4. *Computer Laboratory*
(5–10 minutes from now...)

