

35007

Integral-Defined Functions

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Formal Function Definitions

Summary of the Story So Far

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!},$$

$$\cos(x) = \sin\left(x + \frac{\pi}{2}\right), \quad \tan(x) = \frac{\sin(x)}{\cos(x)},$$

$$\ln(x) = \int_1^x \frac{1}{t} dt, \quad \exp(x) = \ln^{-1}(x).$$

Examples of Other Integral-Defined Functions

Airy Function of the First Kind

$$\text{Ai}(x) = \frac{1}{\pi} \int_0^{\infty} \cos \left(\frac{t^3}{3} + xt \right) dt, \quad x \in \mathbb{R}.$$

Error Function

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \quad x \in \mathbb{R}.$$

Exponential Integral Function

$$\text{Ei}(x) = \int_{-\infty}^x \frac{e^t}{t} dt, \quad x \neq 0.$$

In contemporary mathematics there are dozens of functions defined via integrals or differential equations.

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Very thick books such as:

Abramowitz and Stegun and

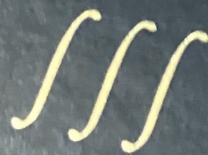
Gradshteyn and Ryzhik

define and give results for these special functions.

Table of Integrals, Series, and Products

Fifth Edition

I. S. Gradshteyn/I. M. Ryzhik
Alan Jeffrey, Editor



Integral-Defined Functions in 35007

We will cover the

Gamma Function

and the

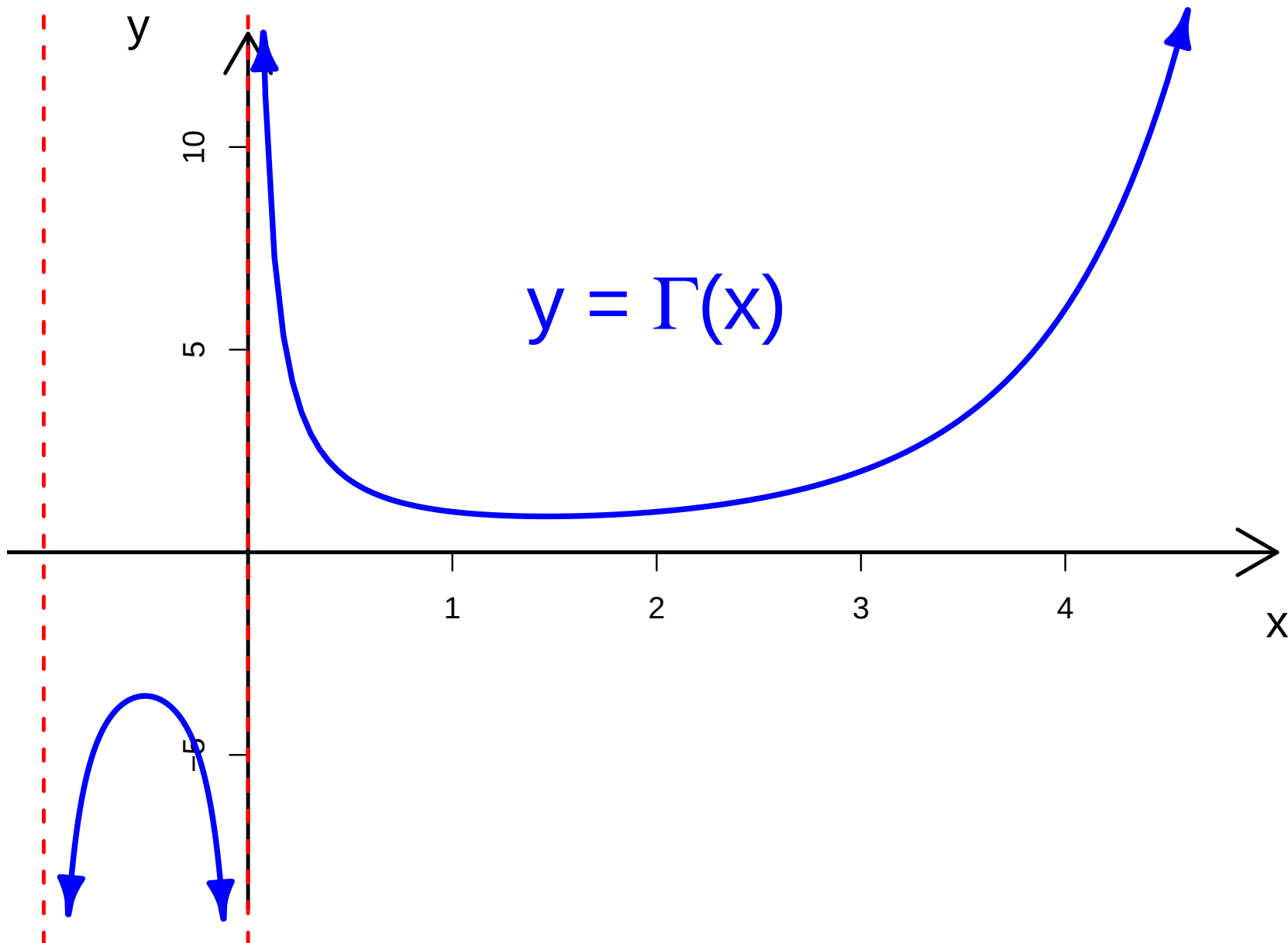
Beta Function

The Gamma Function

Definition of the Gamma Function

For $x > -1, x \neq 0$:

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt.$$



Gamma Function Results

$$\Gamma(n) = (n - 1)!, \quad n \in \mathbb{N},$$

$$\Gamma(x + 1) = x\Gamma(x), \quad x > 0,$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Gamma Function Examples

$$\Gamma(5) = 4! = 24$$

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How about $\Gamma(5 \frac{1}{2})$?

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How about $\Gamma(5 \frac{1}{2})$?

$$\Gamma(5 \frac{1}{2}) = \Gamma(4 \frac{1}{2} + 1) = (4 \frac{1}{2}) \Gamma(4 \frac{1}{2})$$

$$= (4 \frac{1}{2})(3 \frac{1}{2}) \Gamma(3 \frac{1}{2}) = (4 \frac{1}{2})(3 \frac{1}{2})(2 \frac{1}{2}) \Gamma(2 \frac{1}{2})$$

$$= (4 \frac{1}{2})(3 \frac{1}{2})(2 \frac{1}{2})(1 \frac{1}{2}) \Gamma(1 \frac{1}{2}) = (4 \frac{1}{2})(3 \frac{1}{2})(2 \frac{1}{2})(1 \frac{1}{2})(\frac{1}{2}) \Gamma(\frac{1}{2})$$

$$= \frac{945\sqrt{\pi}}{32}$$

Gamma Function Integration by Substitution Example

Wanted:

$$I = \int_0^{\infty} e^{-23x^5} dx$$

SOLUTION: (to $I = \int_0^\infty e^{-23x^5} dx$)

$$u = 23x^5 \implies x = \frac{u^{1/5}}{23^{1/5}} \implies \frac{dx}{du} = \frac{u^{-4/5}}{5(23^{1/5})}$$
$$\implies dx = \frac{u^{-4/5}}{5(23^{1/5})} du$$

Also, $x = 0 \implies u = 0$ and $x = \infty \implies u = \infty$.

Hence,

$$I = \frac{1}{5(23^{1/5})} \int_0^\infty u^{-4/5} e^{-u} du = \frac{1}{5(23^{1/5})} \int_0^\infty u^{1/5-1} e^{-u} du$$
$$= \frac{\Gamma(\frac{1}{5})}{5(23^{1/5})}$$

ANSWER: $I = \frac{\Gamma(\frac{1}{5})}{5(23^{1/5})}.$

The Beta Function

Definition of the Beta Function

For $x, y > 0$:

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

Beta Function Main Result

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}$$

Beta Function Example

Wanted:

$$J = \int_0^1 x^4 (1 - x)^7 dx$$

SOLUTION: (to $J = \int_0^1 x^4(1-x)^7 dx$)

$$J = \int_0^1 x^{5-1}(1-x)^{8-1} dx$$

$$= B(5, 8) = \frac{\Gamma(5)\Gamma(8)}{\Gamma(13)} = \frac{(4!)(7!)}{12!}$$

$$= \frac{1}{3960}$$

ANSWER: $J = \frac{1}{3960}.$

The Most Relevant Exercises
(from the subject exercise sets)

L.1., L.2.

