35007

Formal Definitions in Real Analysis

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What is Two Cubed?

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$$

How about these:

$$2^{3.1}$$

$$2^{\sqrt{7}}$$

$$2^{\pi} \quad \textbf{?}$$

What is Two Cubed?

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\begin{aligned}
2^3 &= 2 \times (2 \times 2) \\
&= (2 \times 2) + (2 \times 2) \\
&= (2 + 2) + (2 + 2) \\
&= 8
\end{aligned}
$$

Ingredients needed to formally define $2^{3.1}$, $2^{\sqrt{7}}$, 2^{π}:

1. $\inf A$ ***and*** $\sup A$.
2. ***Sequences.***
3. ***Limits.***
4. ***Function Inverses.***

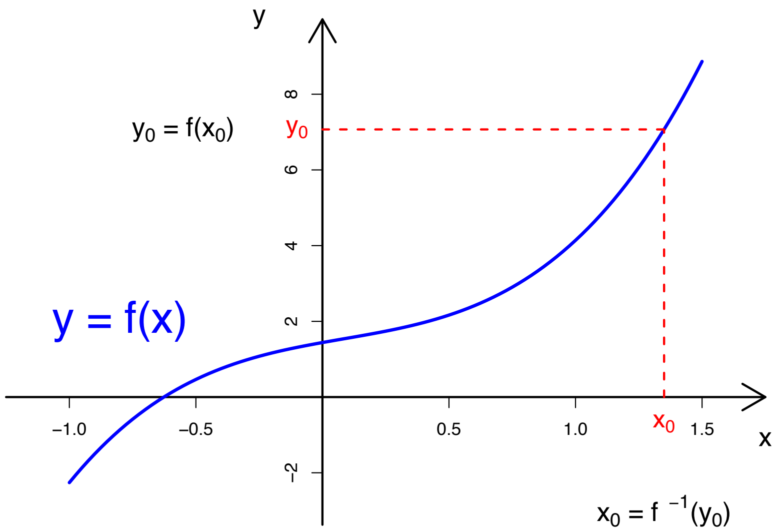
Function Inverses

Function Inverses in General

$f : A \to B$ for general sets A and B where f is one to one.

$f^{-1} : B \to A$ and for each $y \in B$

$$f^{-1}(y) = \Big[x \in A \text{ such that } y = f(x)\Big].$$



Function Inverses Example 1

f_1 : Australian states $\to$ Australian state capitals is given by:

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Example:

$$\begin{aligned} f_1^{-1}(\text{Hobart}) &= \left[x \in \text{Australian states such that Hobart} = f_1(x)\right] \\ &= \left[x \in \text{Australian states such that Hobart} = \text{capital of } x\right] \\ &= \text{Tasmania} \end{aligned}$$

Example solution: $f_1^{-1}(\text{Hobart}) = \text{Tasmania}$.

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Function Inverses Example 2

$f_2 : \mathbb{R} \to \mathbb{R}$ is given by: $f_2(x) = 4x - 23$.

$$\begin{aligned} f_2^{-1}(y) &= \left[x \in \mathbb{R} \text{ such that } y = f_2(x)\right] \\ &= \left[x \in \mathbb{R} \text{ such that } y = 4x - 23\right] \\ &= \left[x \in \mathbb{R} \text{ such that } y + 23 = 4x\right] \\ &= \left[x \in \mathbb{R} \text{ such that } x = (y+23)/4\right] \\ &= (y+23)/4. \end{aligned}$$

For real number domain and range we can use a direct algebra approach ...

$$
\begin{aligned}
y = f_2(x) \quad &\iff \quad y = 4x - 23 \\
&\iff \quad y + 23 = 4x \\
&\iff \quad x = (y+23)/4
\end{aligned}
$$

Therefore, $\quad f_2^{-1}(y) = (y+23)/4.$

Function Inverses Class Exercise

If $f_3 : \mathbb{R} \to \mathbb{R}$ is given by

$$f_3(x) = \frac{\sqrt[9]{x} + 117}{38}$$

then what is f_3^{-1}?

Solution

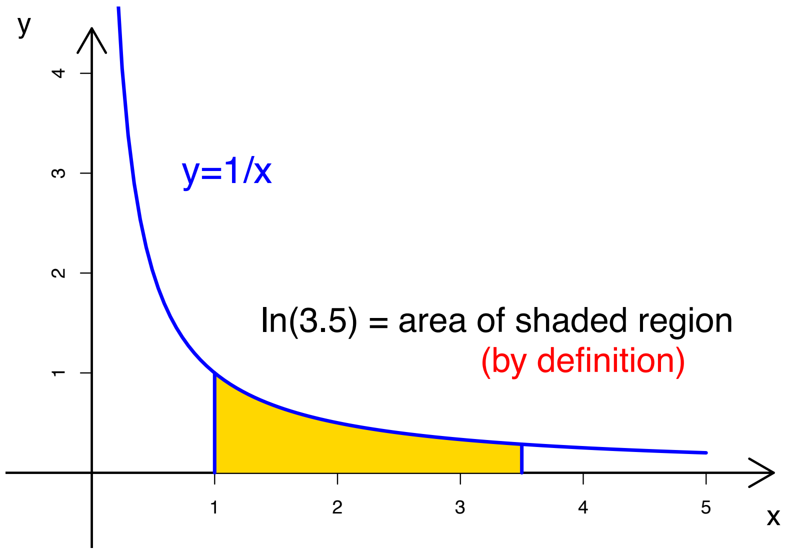
$$
\begin{aligned}
y = f_3(x) \quad &\iff \quad y = \left(\sqrt[9]{x} + 117\right)/38 \\
&\iff \quad 38y = \sqrt[9]{x} + 117 \\
&\iff \quad 38y - 117 = \sqrt[9]{x} \\
&\iff \quad x = (38y - 117)^9.
\end{aligned}
$$

Therefore, $\quad f_3^{-1}(y) = (38y - 117)^9.$

Definition

For each $x > 0$, the natural logarithm function at x is defined to be

$$\ln(x) = \int_1^x \frac{1}{t}\, dt.$$



Definition

For each $x > 0$, the natural logarithm function at x is defined to be

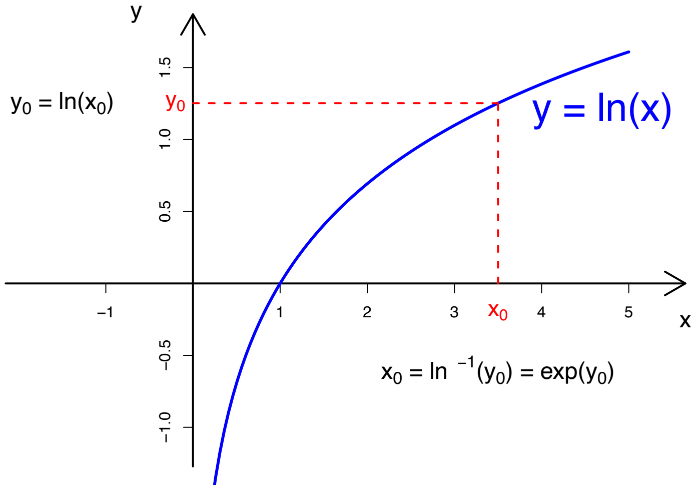
$$\ln(x) = \int_1^x \frac{1}{t}\, dt.$$

The precise meaning of the right-hand side is the topic of Chapter 4 of the notes (Riemann Integration).

Definition

For each $x \in \mathbb{R}$, the exponential function at x is defined to be

$$\exp(x) = \ln^{-1}(x).$$



The Formal Real Analysis
Definition of 2^3

$$2^3 = \exp\Big(3\ln(2)\Big)$$

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$$= \left[x > 0 \text{ such that } \int_1^x \frac{1}{t}\,dt = 3\int_1^2 \frac{1}{t}\,dt\right].$$

The Formal Real Analysis Definition of 2^π

$$2^\pi = \exp\Big(\pi\ln(2)\Big)$$

$$= \ln^{-1}\Big(\pi\ln(2)\Big)$$

$$= \left[x > 0 \text{ such that } \int_1^x \frac{1}{t}\,dt = \pi\int_1^2 \frac{1}{t}\,dt\right].$$

The Formal Real Analysis Definition of $2^{\sqrt{7}}$

$$2^{\sqrt{7}} = \exp\Big(\sqrt{7}\ln(2)\Big)$$

$$= \ln^{-1}\Big(\sqrt{7}\ln(2)\Big)$$

$$= \left[x > 0 \text{ such that } \int_1^x \frac{1}{t}\,dt = \sqrt{7}\int_1^2 \frac{1}{t}\,dt\right].$$

But what is π?

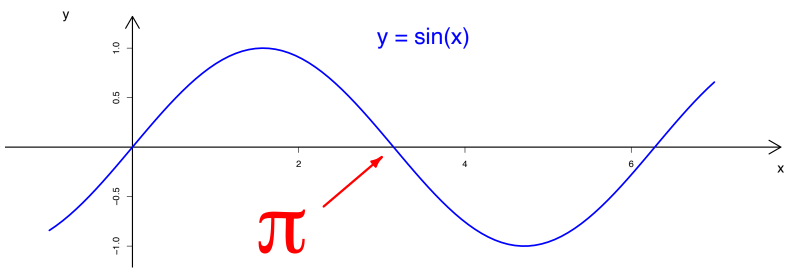
Definition

$$\pi = \inf\{x > 0 : \sin(x) = 0\}$$

And what is $\sin(x)$?

Definition

$$\pi = \inf\{x > 0 : \sin(x) = 0\}$$



Definition

For each $x \in \mathbb{R}$ the sine function at x is defined to be

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Definition

For each $x \in \mathbb{R}$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Hence

$$\pi = \inf\left\{ x > 0 : \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = 0 \right\}.$$

Formal Definitions in Real Analysis

2^3 is ***NOT defined to be*** $2 \times 2 \times 2$.

$\sin(x)$ is ***NOT defined*** using triangles.

π is ***NOT defined*** using circles.

Formal Definitions of Other Trigonometric Functions

$$\cos(x) = \sin\left(x + \frac{\pi}{2}\right).$$

Also

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \quad \sec(x) = \frac{1}{\cos(x)} \quad \text{for } \cos(x) \neq 0$$

and

$$\cot(x) = \frac{\cos(x)}{\sin(x)}, \quad \operatorname{cosec}(x) = \frac{1}{\sin(x)} \quad \text{for } \sin(x) \neq 0.$$

Avoiding Circularity

The founding fathers and mothers of real analysis have figured out that weird definitions such as

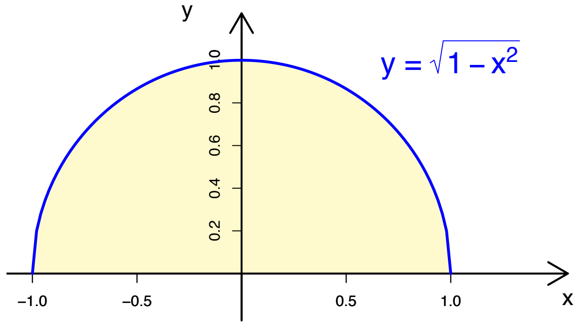
$$2^3 = \left[x > 0 \text{ such that } \int_1^x \frac{1}{t}\,dt = 3\int_1^2 \frac{1}{t}\,dt \right].$$

are needed to avoid circularity and have everything being nice and consistent.

Matching the Physical World

Thankfully, the formal definitions still lead to $2^3 = 8$ and

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \text{area of the unit semi-circle} = \frac{\pi}{2}.$$



The Most Relevant Exercise
(from the current subject exercise sets)

A.4

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