

35007

***Formal
Definitions in
Real Analysis***

Professor Matt Wand

University of Technology Sydney

What is Two Cubed?

What is Two Cubed?

$$2^3 = 2 \times (2 \times 2)$$

What is Two Cubed?

$$\begin{aligned} 2^3 &= 2 \times (2 \times 2) \\ &= (2 \times 2) + (2 \times 2) \end{aligned}$$

What is Two Cubed?

$$\begin{aligned}2^3 &= 2 \times (2 \times 2) \\&= (2 \times 2) + (2 \times 2) \\&= (2 + 2) + (2 + 2)\end{aligned}$$

What is Two Cubed?

$$\begin{aligned}2^3 &= 2 \times (2 \times 2) \\&= (2 \times 2) + (2 \times 2) \\&= (2 + 2) + (2 + 2) \\&= 8\end{aligned}$$

How about these:

$$2^{3.1}$$

$$2^{\sqrt{7}}$$

$$2^{\pi}$$

?

***Ingredients needed to
formally define $2^{3.1}$, $2^{\sqrt{7}}$, 2^{π} :***

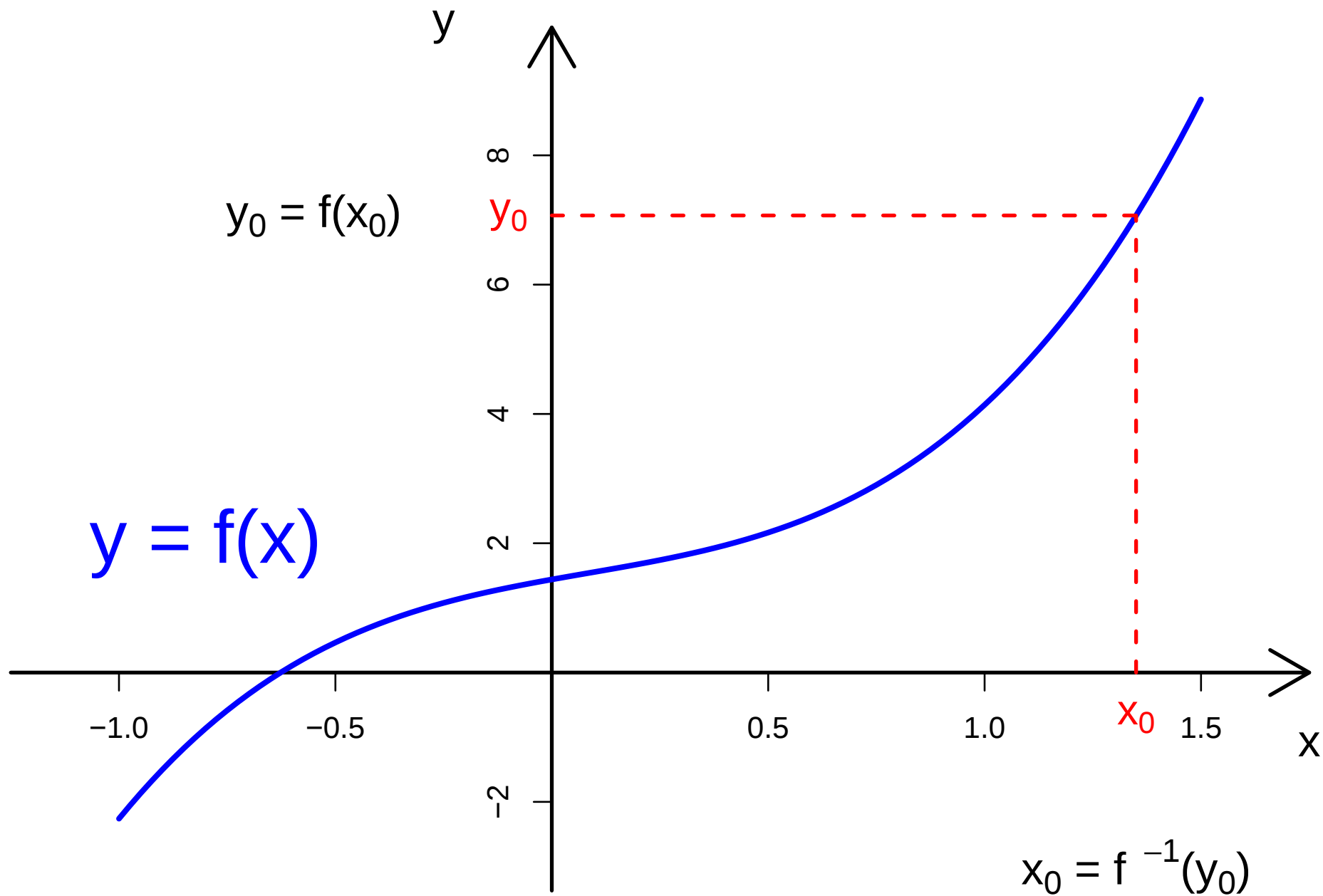
1. $\inf A$ ***and*** $\sup A$.

2. ***Sequences.***

3. ***Limits.***

4. ***Function Inverses.***

Function Inverses



Function Inverses in General

$f : A \rightarrow B$ for general sets A and B
where f is one to one.

$f^{-1} : B \rightarrow A$ and for each $y \in B$

$$f^{-1}(y) = \left[x \in A \text{ such that } y = f(x) \right].$$

Function Inverses Example 1

f_1 : Australian states $\rightarrow$ Australian state capitals is given by:

$$f_1(x) = \text{capital of } x.$$

Function Inverses Example 1

f_1 : Australian states $\rightarrow$ Australian state capitals is given by:

$$f_1(x) = \text{capital of } x.$$

f_1^{-1} : Australian state capitals $\rightarrow$ Australian states and

$$f_1^{-1}(y) = \left[x \in \text{Australian states such that } y = f_1(x) \right]$$

Function Inverses *Example 1*

f_1 : Australian states $\rightarrow$ Australian state capitals is given by:

$$f_1(x) = \text{capital of } x.$$

f_1^{-1} : Australian state capitals $\rightarrow$ Australian states and

$$f_1^{-1}(y) = \left[x \in \text{Australian states such that } y = f_1(x) \right]$$

Example:

$$\begin{aligned} f_1^{-1}(\text{Hobart}) &= \left[x \in \text{Australian states such that Hobart} = f_1(x) \right] \\ &= \left[x \in \text{Australian states such that Hobart} = \text{capital of } x \right] \\ &= \text{Tasmania} \end{aligned}$$

Function Inverses *Example 1*

f_1 : Australian states $\rightarrow$ Australian state capitals is given by:

$$f_1(x) = \text{capital of } x.$$

f_1^{-1} : Australian state capitals $\rightarrow$ Australian states and

$$f_1^{-1}(y) = \left[x \in \text{Australian states such that } y = f_1(x) \right]$$

Example:

$$\begin{aligned} f_1^{-1}(\text{Hobart}) &= \left[x \in \text{Australian states such that } \text{Hobart} = f_1(x) \right] \\ &= \left[x \in \text{Australian states such that } \text{Hobart} = \text{capital of } x \right] \\ &= \text{Tasmania} \end{aligned}$$

Example solution: $f_1^{-1}(\text{Hobart}) = \text{Tasmania}.$

Function Inverses *Example 2*

$f_2 : \mathbb{R} \rightarrow \mathbb{R}$ is given by: $f_2(x) = 4x - 23$.

$$\begin{aligned} f_2^{-1}(y) &= \left[x \in \mathbb{R} \text{ such that } y = f_2(x) \right] \\ &= \left[x \in \mathbb{R} \text{ such that } y = 4x - 23 \right] \\ &= \left[x \in \mathbb{R} \text{ such that } y + 23 = 4x \right] \\ &= \left[x \in \mathbb{R} \text{ such that } x = (y + 23)/4 \right] \\ &= (y + 23)/4. \end{aligned}$$

***For real number domain
and range we can
use a direct algebra
approach ...***

$$y = f_2(x) \iff y = 4x - 23$$

$$\iff y + 23 = 4x$$

$$\iff x = (y + 23)/4$$

Therefore, $f_2^{-1}(y) = (y + 23)/4$.

Function Inverses Class Exercise

If $f_3 : \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$f_3(x) = \frac{\sqrt[9]{x} + 117}{38}$$

then what is f_3^{-1} ?

Solution

$$y = f_3(x) \iff y = (\sqrt[9]{x} + 117)/38$$

$$\iff 38y = \sqrt[9]{x} + 117$$

$$\iff 38y - 117 = \sqrt[9]{x}$$

$$\iff x = (38y - 117)^9.$$

Therefore, $f_3^{-1}(y) = (38y - 117)^9.$

Definition

For each $x > 0$, the natural logarithm function at x is defined to be

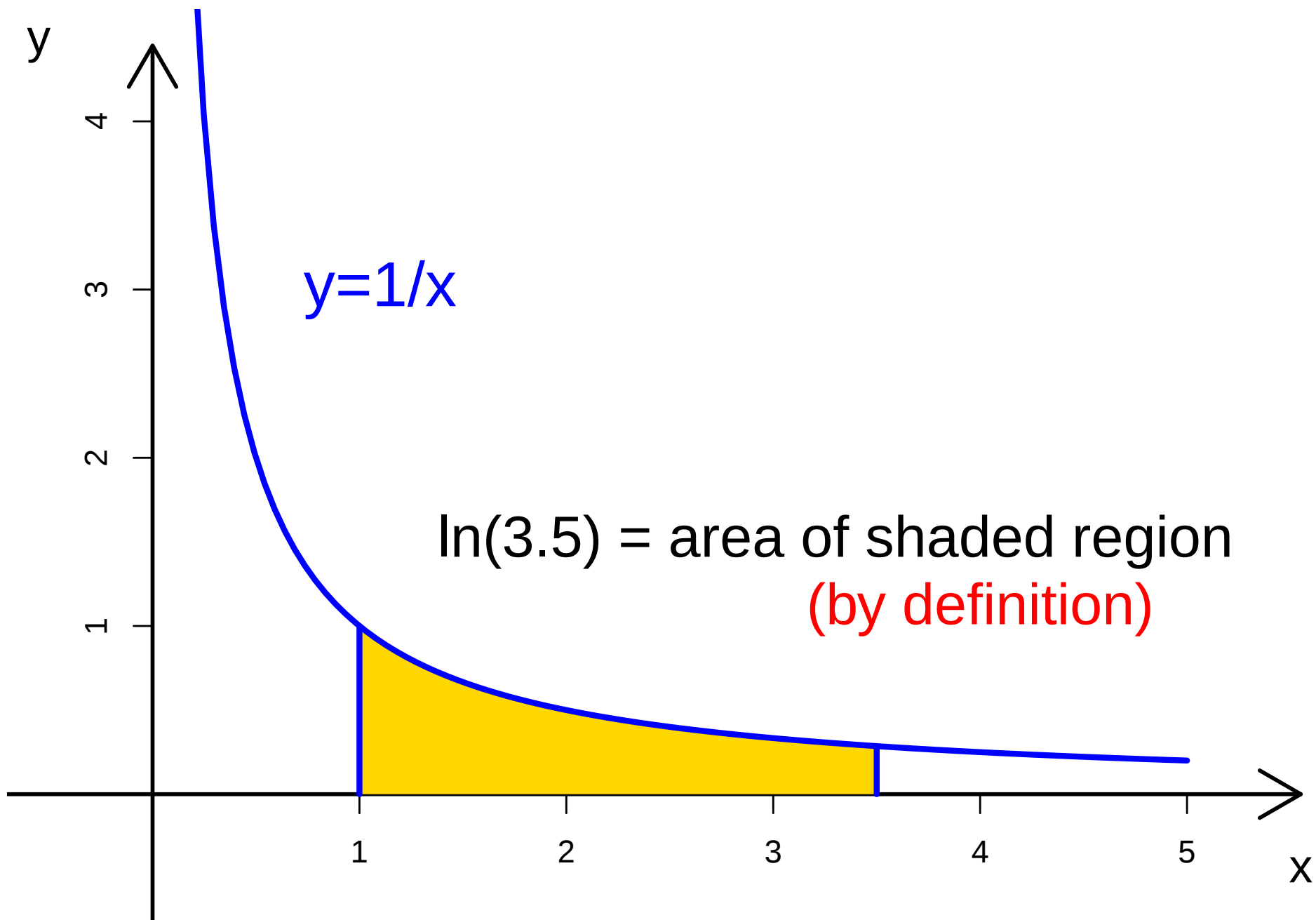
$$\ln(x) = \int_1^x \frac{1}{t} dt.$$

Definition

For each $x > 0$, the natural logarithm function at x is defined to be

$$\ln(x) = \int_1^x \frac{1}{t} dt.$$

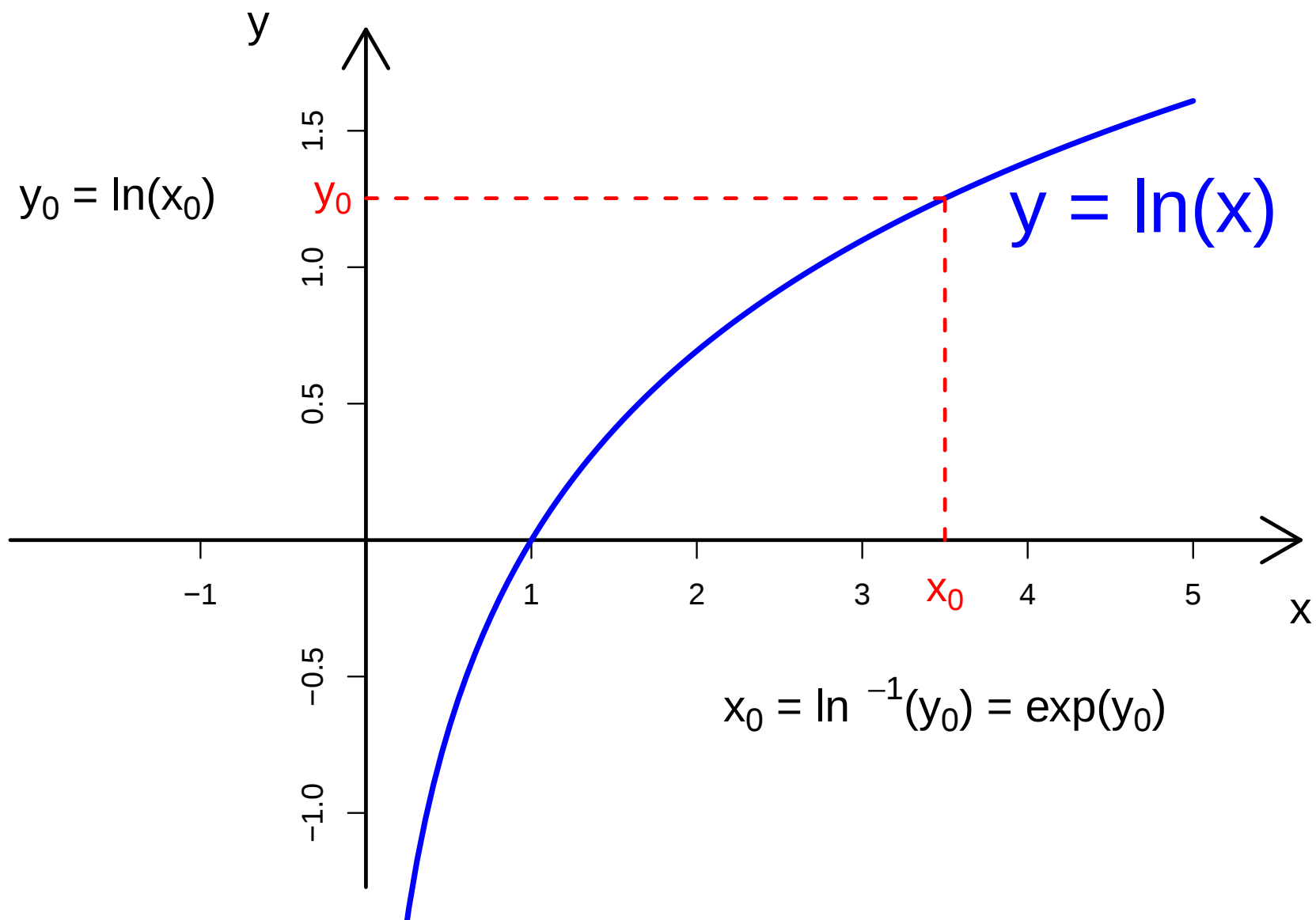
The precise meaning of the right-hand side is the topic of Chapter 4 of the notes (Riemann Integration).



Definition

For each $x \in \mathbb{R}$, the exponential function at x is defined to be

$$\exp(x) = \ln^{-1}(x).$$



The Formal Real Analysis

Definition of 2^3

The Formal Real Analysis

Definition of 2^3

$$2^3 = \exp \left(3 \ln(2) \right)$$

The Formal Real Analysis

Definition of 2^3

$$\begin{aligned} 2^3 &= \exp \left(3 \ln(2) \right) \\ &= \ln^{-1} \left(3 \ln(2) \right) \end{aligned}$$

The Formal Real Analysis

Definition of 2^3

$$2^3 = \exp \left(3 \ln(2) \right)$$

$$= \ln^{-1} \left(3 \ln(2) \right)$$

$$= \left[x > 0 \text{ such that } \int_1^x \frac{1}{t} dt = 3 \int_1^2 \frac{1}{t} dt \right].$$

The Formal Real Analysis

Definition of $2^{\sqrt{7}}$

$$2^{\sqrt{7}} = \exp \left(\sqrt{7} \ln(2) \right)$$

$$= \ln^{-1} \left(\sqrt{7} \ln(2) \right)$$

$$= \left[x > 0 \text{ such that } \int_1^x \frac{1}{t} dt = \sqrt{7} \int_1^2 \frac{1}{t} dt \right].$$

The Formal Real Analysis

Definition of 2^π

$$2^\pi = \exp \left(\pi \ln(2) \right)$$

$$= \ln^{-1} \left(\pi \ln(2) \right)$$

$$= \left[x > 0 \text{ such that } \int_1^x \frac{1}{t} dt = \pi \int_1^2 \frac{1}{t} dt \right].$$

But what is

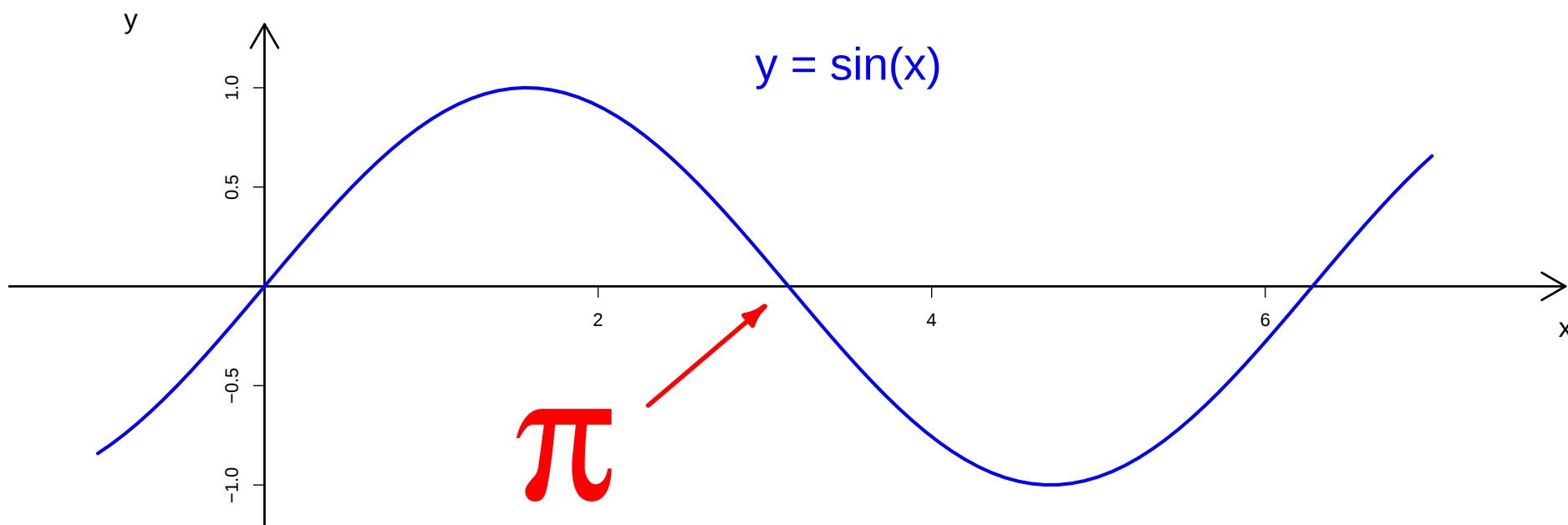
π ?

Definition

$$\pi = \inf\{x > 0 : \sin(x) = 0\}$$

Definition

$$\pi = \inf\{x > 0 : \sin(x) = 0\}$$



And what is

$\sin(x)$?

Definition

For each $x \in \mathbb{R}$ the **sine** function at x is **defined** to be

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Definition

For each $x \in \mathbb{R}$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Hence

$$\pi = \inf \left\{ x > 0 : \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = 0 \right\}.$$

Formal Definitions of Other Trigonometric Functions

$$\cos(x) = \sin\left(x + \frac{\pi}{2}\right).$$

Also

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \quad \sec(x) = \frac{1}{\cos(x)} \quad \text{for } \cos(x) \neq 0$$

and

$$\cot(x) = \frac{\cos(x)}{\sin(x)}, \quad \operatorname{cosec}(x) = \frac{1}{\sin(x)} \quad \text{for } \sin(x) \neq 0.$$

Formal Definitions in Real Analysis

2^3 is **NOT** defined to be $2 \times 2 \times 2$.

$\sin(x)$ is **NOT** defined using triangles.

π is **NOT** defined using circles.

Avoiding Circularity

The founding fathers and mothers of real analysis have figured out that weird definitions such as

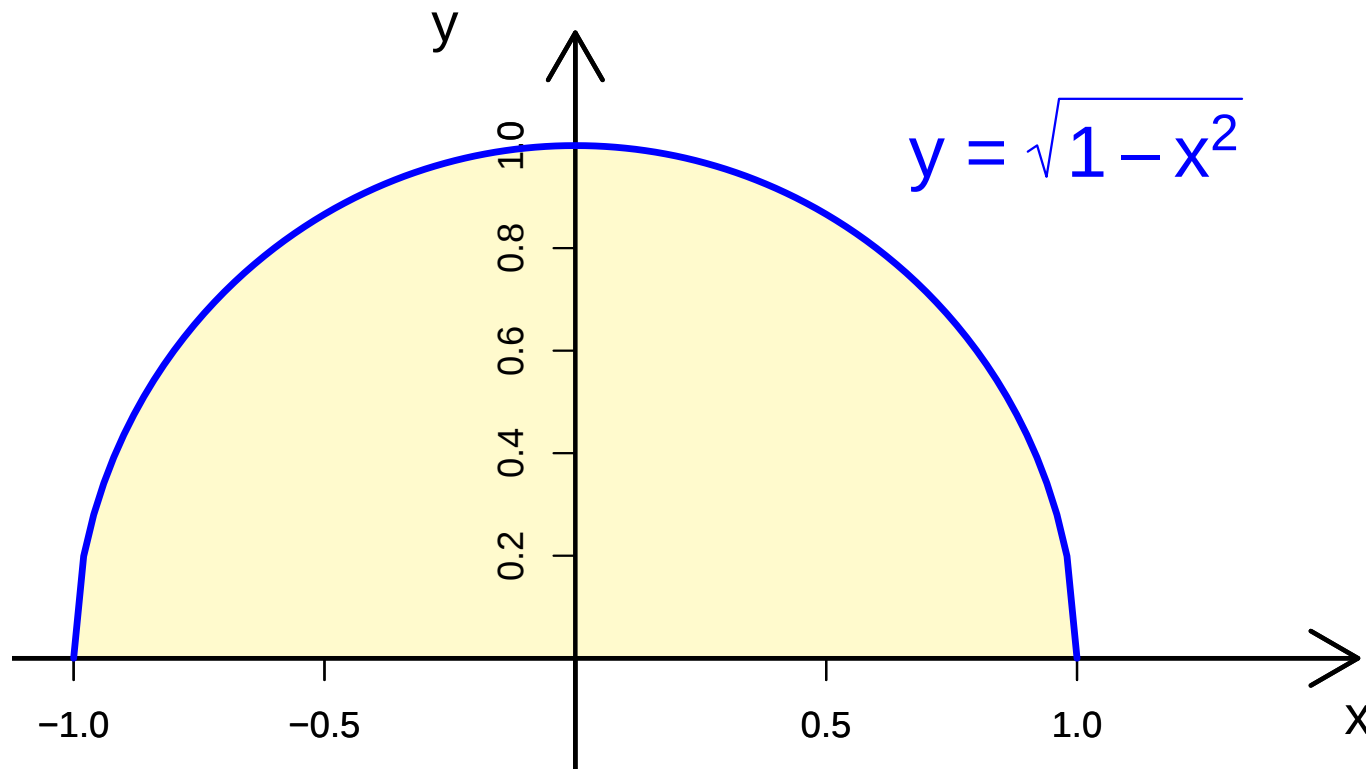
$$2^3 = \left[x > 0 \text{ such that } \int_1^x \frac{1}{t} dt = 3 \int_1^2 \frac{1}{t} dt \right].$$

are needed to **avoid circularity** and have everything being nice and **consistent**.

Matching the Physical World

Thankfully, the formal definitions still lead to $2^3 = 8$ and

$$\int_{-1}^1 \sqrt{1-x^2} dx = \text{area of the unit semi-circle} = \frac{\pi}{2}.$$



The Most Relevant Exercise
(from the current subject exercise sets)

A.4

