35007

Finishing Chapters 1–3

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Power Series

First Half FINAL Main Topics

1. Power Series
2. Series Evaluation
3. Taylor Expansion
4. Taylor's Theorem
5. L'Hôpital's Rule

Power Series Example

For each $x \in \mathbb{R}$ consider

$$\sum_{n=0}^{\infty} \frac{\sqrt{7n^2 + 2n + 38}\,(x-9)^n}{4^n}.$$

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$$\sum_{n=0}^{\infty} \frac{\sqrt{7n^2+2n+38}\,(x-9)^n}{4^n}.$$

Key Question: For which x does this series converge?

Step 1. Obtain L

If $a_n = \sqrt{7n^2+2n+38}/4^n$ then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\sqrt{7(1+1/n)^2+2(1/n+1/n^2)+38/n^2}}{4\sqrt{7+2/n+38/n^2}}$$

$$\Longrightarrow \qquad L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{4}.$$

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$$\Longrightarrow \qquad L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{4}.$$

Step 2. Obtain region of convergence

[Definition 3.28, Theorem 3.30]

Series converges for

$$|x-9|L < 1 \quad \Longleftrightarrow \quad |x-9| < 4 \quad \Longleftrightarrow \quad 5 < x < 13.$$

Step 3. Check the boundaries

$$x = 5 \quad \longrightarrow \sum_{n=0}^{\infty} \sqrt{7n^2+2n+38}\,(-1)^n$$ ***DIVERGES***

$$x = 13 \quad \longrightarrow \sum_{n=0}^{\infty} \sqrt{7n^2+2n+38}$$ ***DIVERGES***

[both by Lemma 1.36]

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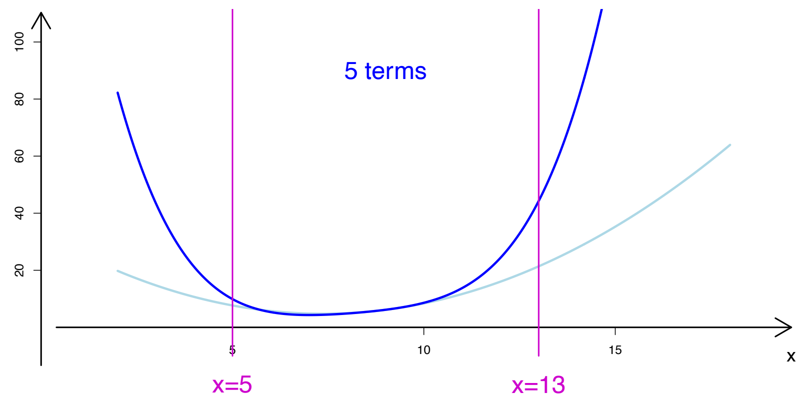
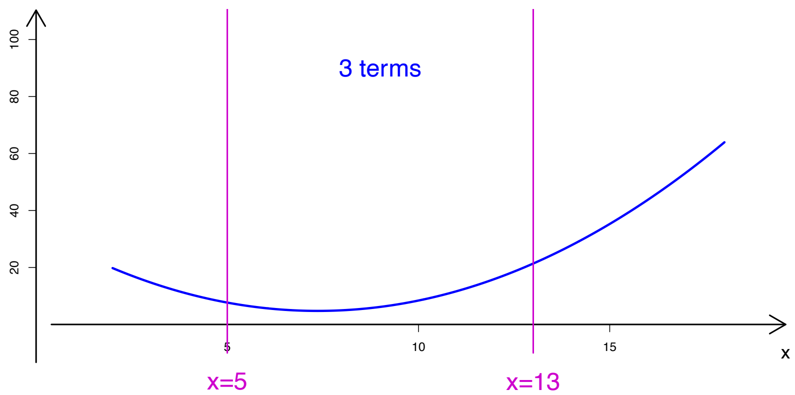
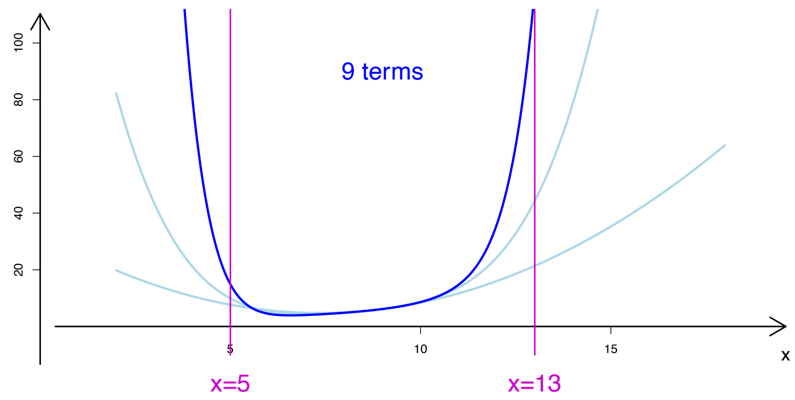
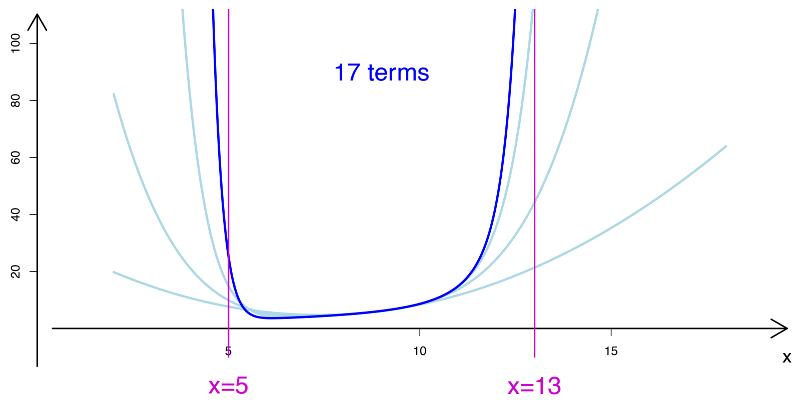


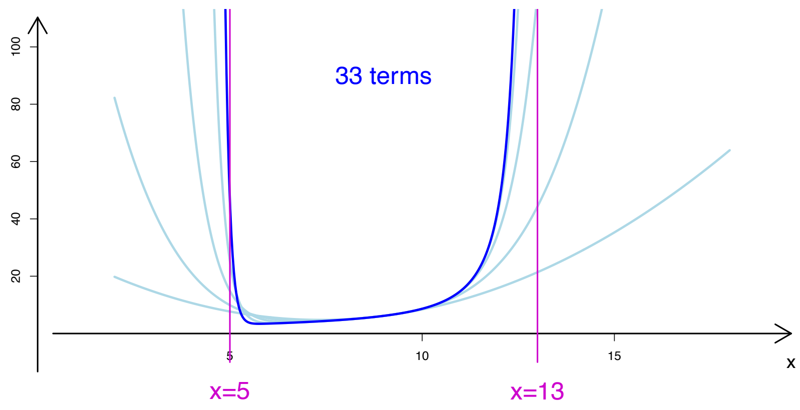
FINAL ANSWER

$$\sum_{n=0}^{\infty} \frac{\sqrt{7n^2 + 2n + 38}\,(x-9)^n}{4^n} \text{converges} \iff 5 < x < 13.$$









Series Evaluation

Series Evaluation Example

Evaluate

$$\frac{5^2}{7} + \frac{6^2}{7^2} + \frac{7^2}{7^3} + \frac{8^2}{7^4} + \dots$$

$$= \frac{25}{7} + \frac{36}{49} + \frac{49}{343} + \frac{64}{2401} + \dots$$

Answer

$$\frac{5^2}{7}+\frac{6^2}{7^2}+\frac{7^2}{7^3}+\frac{8^2}{7^4}+\ldots = 4\,\frac{13}{27}$$ **EXACTLY**

Derivation, Part 1

$$\frac{5^2}{7}+\frac{6^2}{7^2}+\frac{7^2}{7^3}+\frac{8^2}{7^4}+\ldots$$

$$= 7^4\left(\frac{5^2}{7^5}+\frac{6^2}{7^6}+\frac{7^2}{7^7}+\frac{8^2}{7^8}+\ldots\right)$$

$$= 7^4\left(\frac{0^2}{7^0}+\frac{1^2}{7^1}+\frac{2^2}{7^2}+\frac{3^2}{7^3}+\frac{4^2}{7^4}+\frac{5^2}{7^5}+\frac{6^2}{7^6}+\frac{7^2}{7^7}+\ldots\right)$$

$$-7^4\left(\frac{0^2}{7^0}+\frac{1^2}{7^1}+\frac{2^2}{7^2}+\frac{3^2}{7^3}+\frac{4^2}{7^4}\right)$$

$$= 7^4\left(\sum_{n=0}^{\infty} n^2x^n\right) - \{(1)(7^3)+(4)(7^2)+(9)7+16\}$$

where $x = 1/7$.

Answer

$$\frac{5^2}{7}+\frac{6^2}{7^2}+\frac{7^2}{7^3}+\frac{8^2}{7^4}+\ldots = 4\,\frac{13}{27}$$ **EXACTLY**

How do we show this?

Derivation, Part 2

Therefore:

$$\frac{5^2}{7}+\frac{6^2}{7^2}+\frac{7^2}{7^3}+\frac{8^2}{7^4}+\ldots = 2401\left(\sum_{n=0}^{\infty} n^2x^n\right)-618$$ where $x = 1/7$.

Derivation, Part 2

Therefore:

$$\frac{5^2}{7}+\frac{6^2}{7^2}+\frac{7^2}{7^3}+\frac{8^2}{7^4}+\ldots = 2401\left(\sum_{n=0}^{\infty} n^2x^n\right)-618 \quad \text{where } x = 1/7.$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \Longrightarrow \quad \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} nx^{n-1} = \frac{d}{dx}\left(\frac{1}{1-x}\right).$$

[*Theorem 3.32*]

Derivation, Part 2

Therefore:

$$\frac{5^2}{7}+\frac{6^2}{7^2}+\frac{7^2}{7^3}+\frac{8^2}{7^4}+\ldots = 2401\left(\sum_{n=0}^{\infty} n^2x^n\right)-618 \quad \text{where } x = 1/7.$$

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[*Theorem 3.32*]

Further maths like this leads to

$$\sum_{n=0}^{\infty} n^2x^n = \frac{x(x+1)}{(1-x)^3}, \quad |x| < 1$$

and the rest is easy.

The Geometric Series Differentiation Trick

This trick allows us to do EXACT evaluation of any

(polynomial in n)$\times r^n$, $|r| < 1$ series.

such as

$$\sum_{n=34}^{\infty} \frac{(87n^9+53n^5-107n^2+65)14^n}{83^n}.$$

Taylor Expansion

Taylor Expansion Example 1

[Definition 3.33] ***Taylor expansion*** of $f(x) = \dfrac{1}{1+x^2}$ about $a = 2$:

$$\frac{1}{1+x^2} = \frac{1}{5} - \frac{4(x-2)}{25} + \frac{11(x-2)^2}{125} - \frac{24(x-2)^3}{625} + \frac{41(x-2)^4}{3125} - \dots$$

which converges for $2-\sqrt{5} < x < 2+\sqrt{5}$.

Taylor Expansion Example 2

[Definition 3.33] ***Taylor expansion*** of $f(x) = e^x \sin(x)$ about $a = \pi$:

$$e^x \sin(x) = -e^{\pi}(x-\pi) - e^{\pi}(x-\pi)^2 - \frac{e^{\pi}(x-\pi)^3}{3} + \frac{e^{\pi}(x-\pi)^5}{30} - \dots$$

which converges for all $x \in \mathbb{R}$.

Taylor Expansion Example 3

[Definition 3.33] ***Taylor expansion*** of $f(x) = \ln(x)$ about $a = 17$:

$$\ln(x) = \ln(17) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-17)^n}{n\,17^n}$$

which converges if and only if

$$0 < x \leq 34.$$

Statistician's Love Taylor Expansion

Taylor Expansion Example 4 (from Statistics)

$X_1, \ldots, X_n$ a random sample from the Normal distribution with mean μ and variance σ^2.

$$\overline{X} = (X_1 + \ldots + X_n)/n = \text{sample mean.}$$

Mean squared error of $e^{\overline{X}} = e^{2\mu}(e^{2\sigma^2/n} - 2e^{\sigma^2/(2n)} + 1)$.

Mean squared error of $e^{\overline{X}} = e^{2\mu}(e^{2\sigma^2/n} - 2e^{\sigma^2/(2n)} + 1)$.

is EXACT but DIFFICULT TO INTERPRET.
Taylor's expansion gives

$$e^{2\sigma^2/n} - 2e^{\sigma^2/(2n)} + 1 = 1 + \frac{2\sigma^2}{n} + \frac{2\sigma^4}{n^2} + \ldots$$
$$-2 - \frac{\sigma^2}{n} - \frac{\sigma^4}{4n^2} + \ldots + 1 \implies$$

Mean squared error of $e^{\overline{X}} = \dfrac{\sigma^2 e^{2\mu}}{n}$+lower order terms

which clearly shows the effect of sample size (n).

Obtaining / Proving Taylor Expansion Convergence Regions Using 35007 Tools

Example 4	(Statistics)	Straightforward via e.g. Definition 3.28, Theorem 3.30
Example 3	$(\ln(x))$	Straightforward via e.g. Definition 3.28, Theorem 3.30
Example 2	$(e^x \sin(x))$	Messier but eventually do-able via notes results
Example 1	$(1/(1+x^2))$	Beyond 35007 (e.g. complex analysis)

Taylor's Theorem

The $n = 1$ version of

Theorem 3.34 (Taylor's Theorem):

Assuming f is "well-behaved" on an open interval containing x and a:

$$f(x) = f(a) + f'(a)(x-a) + \tfrac{1}{2} f''(\xi)(x-a)^2$$

for ξ between a and x.

By Taylor's Theorem (Theorem 3.34) with $a = 0$ and $n = 1$:

$$e^{-x} = 1 - x + \frac{e^{-\xi}x^2}{2} \ (\bigstar) \quad \text{for some} \quad 0 < \xi < x.$$

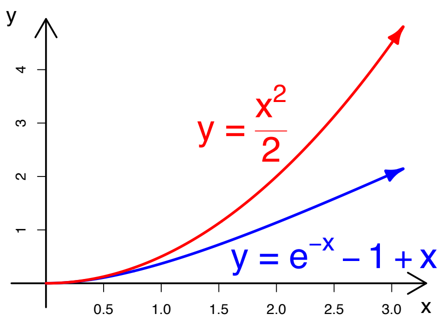
Note that

$$\xi > 0 \implies -\xi < 0 \implies e^{-\xi} < e^0 = 1 \ (\bigstar).$$

Substitution of ($\bigstar$) into ($\bigstar$) gives

$$e^{-x} < 1 - x + \frac{x^2}{2} \iff e^{-x} - x + 1 < \frac{x^2}{2} \text{ for all } x > 0.$$

Taylor's Theorem Example



GOAL: A formal proof that $e^{-x} - x + 1 < \frac{x^2}{2}$ for all $x > 0$.

L'Hôpital's Rule

Our First Function Limits

1. $\lim_{x\to 3} x^2 = 9$
2. $\lim_{x\to 0} \dfrac{\sin(x)}{x} = 1$
3. $\lim_{x\to \infty} \dfrac{\ln(x)}{\ln(x^3+x)} = \dfrac{1}{3}$

2. $\lim_{x\to 0} \dfrac{\sin(x)}{x}$

$$\lim_{x\to 0} \sin(x) = \lim_{x\to 0} x = 0 \qquad \longrightarrow \frac{0}{0}.$$

Using L'Hôpital's Rule [Theorem 3.15]

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}\sin(x)}{\frac{d}{dx}x} = \lim_{x\to 0} \frac{\cos(x)}{1} = 1.$$

3. $\lim_{x\to \infty} \dfrac{\ln(x)}{\ln(x^3+x)}$

$$\lim_{x\to\infty} \ln(x) = \lim_{x\to\infty} \ln(x^3+x) = \infty \qquad \longrightarrow \frac{\infty}{\infty}.$$

Using L'Hôpital's Rule [Theorem 3.15]

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln(x)}{\ln(x^3+x)} &= \lim_{x\to\infty} \frac{\frac{d}{dx}\ln(x)}{\frac{d}{dx}\ln(x^3+x)} \\ &= \lim_{x\to\infty} \frac{1/x}{(3x^2+1)/(x^3+x)} \\ &= \lim_{x\to\infty} \left(\frac{x^3+x}{3x^3+x}\right) \\ &= \lim_{x\to\infty} \left(\frac{1+1/x^2}{3+1/x^2}\right) = \frac{1}{3}.\end{aligned}$$

Position in the Notes

The end of these slides and the Week 5 class

CORRESPONDS TO

the end of Chapter 3 on page 52.

The Most Relevant Exercises
(from the current subject exercise sets)

Exercise Set H

Exercise Set I
excluding I.2, I.8, I.16.

Suggestion if Time Remaining

Halfway Point "Town Hall"- Style Questions and Answers

after a 5 minute break

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