

35007

Finishing
Chapters 1–3

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First Half FINAL Main Topics

1. *Power Series*
2. *Series Evaluation*
3. *Taylor Expansion*
4. *Taylor's Theorem*
5. *L'Hôpital's Rule*

Power Series

Power Series Example

For each $x \in \mathbb{R}$ consider

$$\sum_{n=0}^{\infty} \frac{\sqrt{7n^2 + 2n + 38} (x - 9)^n}{4^n}.$$

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For each $x \in \mathbb{R}$ consider

$$\sum_{n=0}^{\infty} \frac{\sqrt{7n^2 + 2n + 38} (x - 9)^n}{4^n}.$$

Key Question: For which x does this series converge?

Step 1. Obtain L

If $a_n = \sqrt{7n^2 + 2n + 38}/4^n$ then

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\sqrt{7(1 + 1/n)^2 + 2(1/n + 1/n^2) + 38/n^2}}{4\sqrt{7 + 2/n + 38/n^2}}$$

$$\Rightarrow L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{4}.$$

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$$\implies L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{4}.$$

Step 2. Obtain region of convergence

[Definition 3.28, Theorem 3.30]

Series converges for

$$|x-9|L < 1 \iff |x-9| < 4 \iff 5 < x < 13.$$

Step 3. Check the boundaries

$$x = 5 \quad \longrightarrow \quad \sum_{n=0}^{\infty} \sqrt{7n^2 + 2n + 38} (-1)^n \quad \textbf{DIVERGES}$$

$$x = 13 \quad \longrightarrow \quad \sum_{n=0}^{\infty} \sqrt{7n^2 + 2n + 38} \quad \textbf{DIVERGES}$$

[both by [Lemma 1.36](#)]

Step 3. Check the boundaries

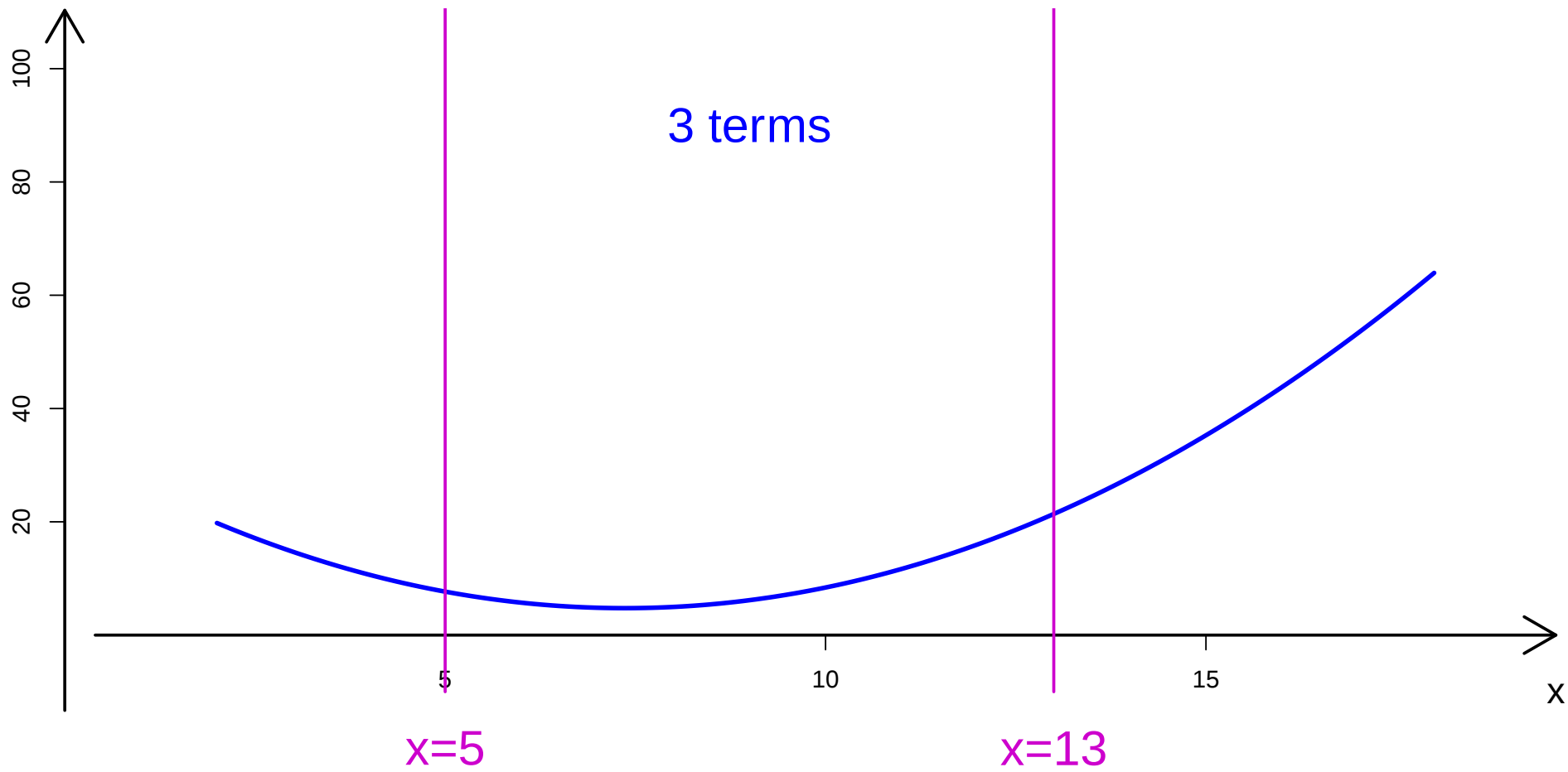
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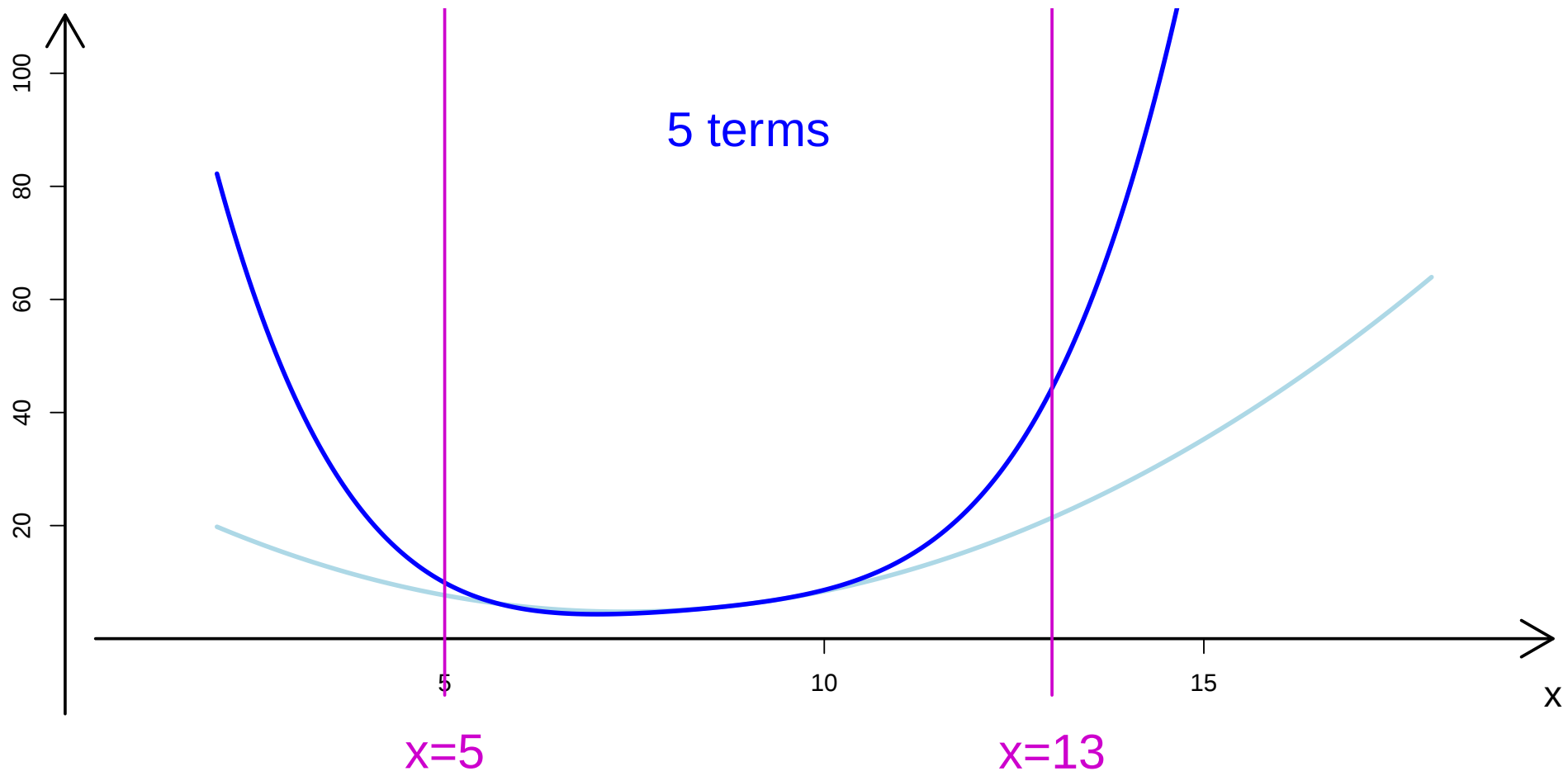
$$x = 13 \quad \longrightarrow \quad \sum_{n=0}^{\infty} \sqrt{7n^2 + 2n + 38} \quad \textbf{DIVERGES}$$

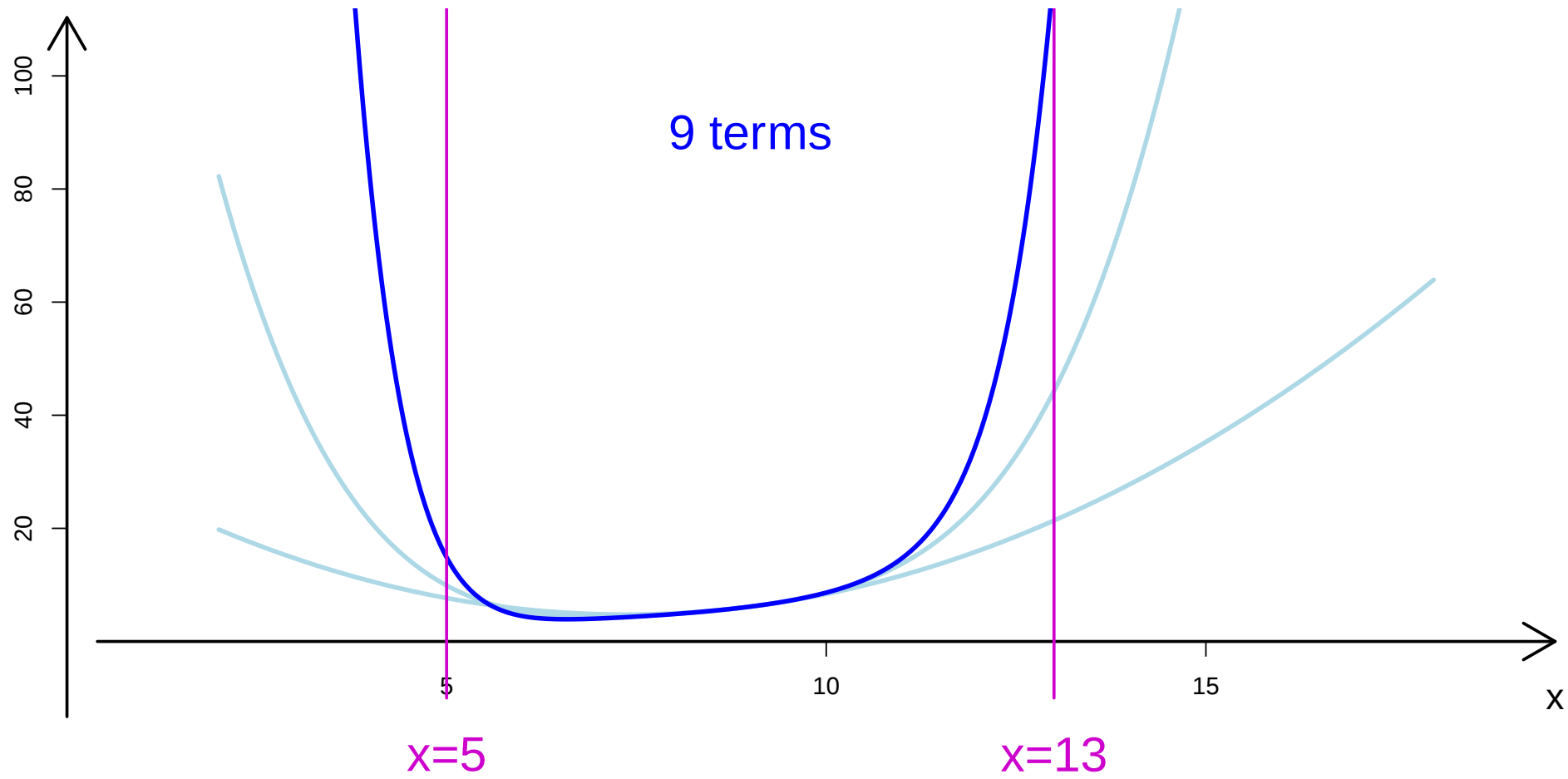
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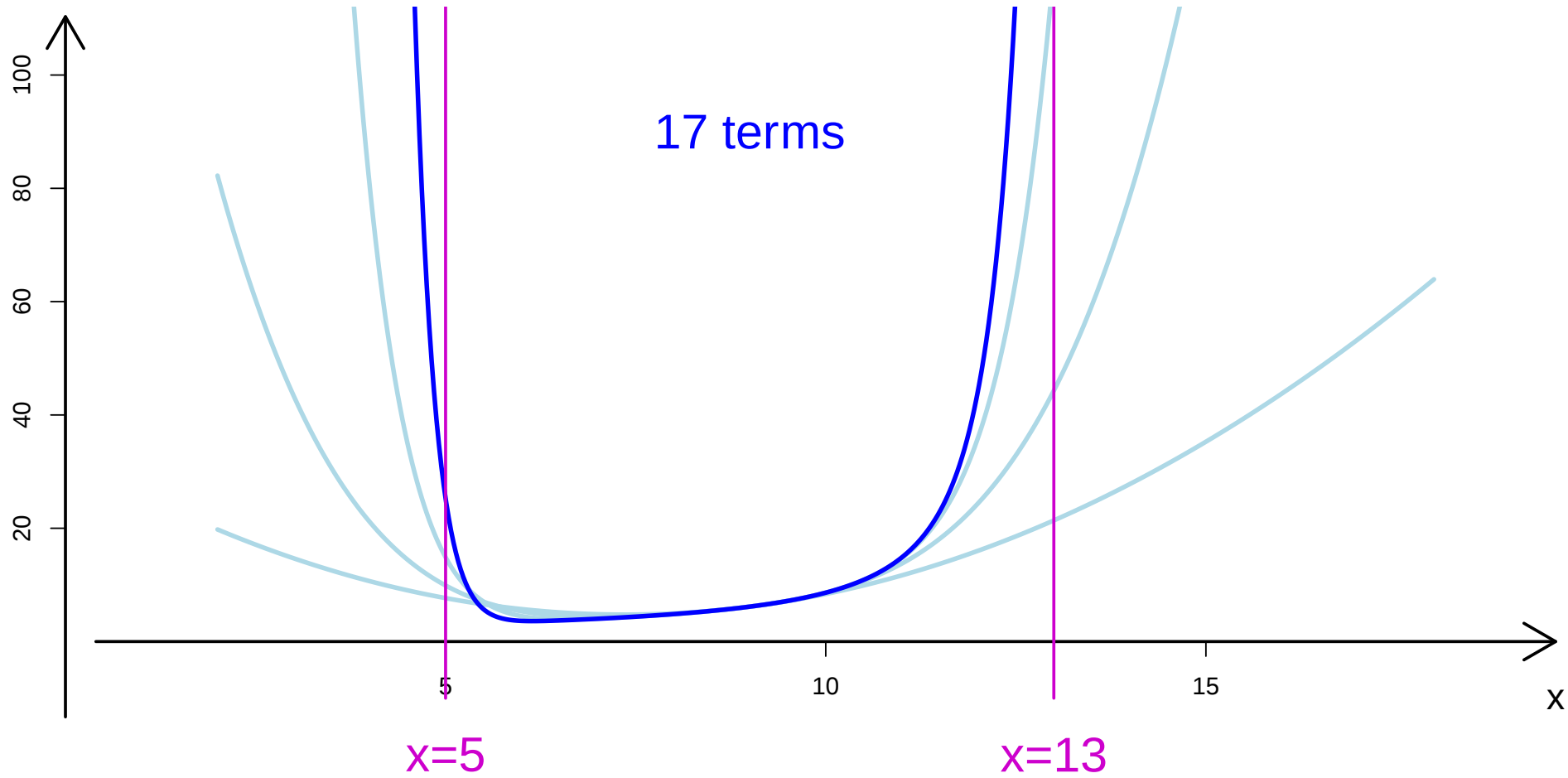
FINAL ANSWER

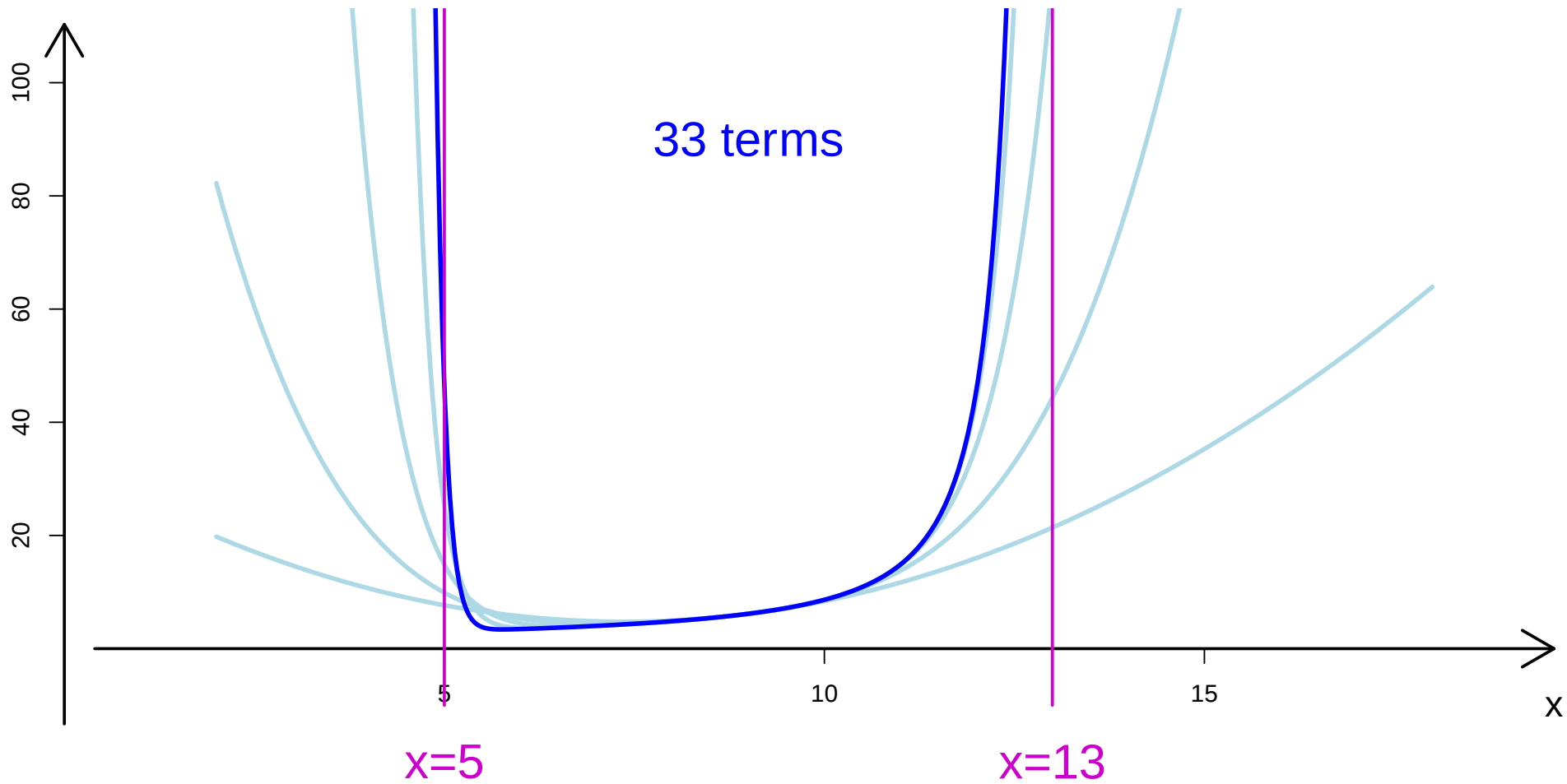
$$\sum_{n=0}^{\infty} \frac{\sqrt{7n^2 + 2n + 38} (x - 9)^n}{4^n} \textbf{converges} \iff 5 < x < 13$$











Series

Evaluation

Series Evaluation Example

Evaluate

$$\frac{5^2}{7} + \frac{6^2}{7^2} + \frac{7^2}{7^3} + \frac{8^2}{7^4} + \dots$$

$$= \frac{25}{7} + \frac{36}{49} + \frac{49}{343} + \frac{64}{2401} + \dots$$

Answer

$$\frac{5^2}{7} + \frac{6^2}{7^2} + \frac{7^2}{7^3} + \frac{8^2}{7^4} + \dots = 4\frac{13}{27} \quad \textit{EXACTLY}$$

Answer

$$\frac{5^2}{7} + \frac{6^2}{7^2} + \frac{7^2}{7^3} + \frac{8^2}{7^4} + \dots = 4\frac{13}{27} \quad \text{EXACTLY}$$

How do we show this?

Derivation, Part 1

$$\frac{5^2}{7} + \frac{6^2}{7^2} + \frac{7^2}{7^3} + \frac{8^2}{7^4} + \dots$$

$$= 7^4 \left(\frac{5^2}{7^5} + \frac{6^2}{7^6} + \frac{7^2}{7^7} + \frac{8^2}{7^8} + \dots \right)$$

$$= 7^4 \left(\frac{0^2}{7^0} + \frac{1^2}{7^1} + \frac{2^2}{7^2} + \frac{3^2}{7^3} + \frac{4^2}{7^4} + \frac{5^2}{7^5} + \frac{6^2}{7^6} + \frac{7^2}{7^7} + \dots \right)$$

$$- 7^4 \left(\frac{0^2}{7^0} + \frac{1^2}{7^1} + \frac{2^2}{7^2} + \frac{3^2}{7^3} + \frac{4^2}{7^4} \right)$$

$$= 7^4 \left(\sum_{n=0}^{\infty} n^2 x^n \right) - \{ (1)(7^3) + (4)(7^2) + (9)7 + 16 \}$$

where $x = 1/7$.

Derivation, Part 2

Therefore:

$$\frac{5^2}{7} + \frac{6^2}{7^2} + \frac{7^2}{7^3} + \frac{8^2}{7^4} + \dots = 2401 \left(\sum_{n=0}^{\infty} n^2 x^n \right) - 618 \quad \text{where } x = 1/7.$$

Derivation, Part 2

Therefore:

$$\frac{5^2}{7} + \frac{6^2}{7^2} + \frac{7^2}{7^3} + \frac{8^2}{7^4} + \dots = 2401 \left(\sum_{n=0}^{\infty} n^2 x^n \right) - 618 \quad \text{where } x = 1/7.$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \Rightarrow \quad \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} n x^{n-1} = \frac{d}{dx} \left(\frac{1}{1-x} \right).$$

[Theorem 3.32]

Derivation, Part 2

Therefore:

$$\frac{5^2}{7} + \frac{6^2}{7^2} + \frac{7^2}{7^3} + \frac{8^2}{7^4} + \dots = 2401 \left(\sum_{n=0}^{\infty} n^2 x^n \right) - 618 \quad \text{where } x = 1/7.$$

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[Theorem 3.32]

Further maths like this leads to

$$\sum_{n=0}^{\infty} n^2 x^n = \frac{x(x+1)}{(1-x)^3}, \quad |x| < 1 \quad \text{and the rest is easy.}$$

The Geometric Series Differentiation Trick

*This **trick** allows us to do **EXACT**
evaluation of any*

(polynomial in n) $\times r^n$, $|r| < 1$ series.

such as

$$\sum_{n=34}^{\infty} \frac{(87n^9 + 53n^5 - 107n^2 + 65)14^n}{83^n}.$$

Taylor Expansion

Taylor Expansion Example 1

[Definition 3.33] *Taylor expansion* of

$f(x) = \frac{1}{1+x^2}$ about $a = 2$:

$$\begin{aligned} \frac{1}{1+x^2} = & \frac{1}{5} - \frac{4(x-2)}{25} + \frac{11(x-2)^2}{125} \\ & - \frac{24(x-2)^3}{625} + \frac{41(x-2)^4}{3125} - \dots \end{aligned}$$

which converges for $2 - \sqrt{5} < x < 2 + \sqrt{5}$.

Taylor Expansion Example 2

[Definition 3.33] *Taylor expansion* of $f(x) = e^x \sin(x)$ about $a = \pi$:

$$e^x \sin(x) = -e^\pi(x - \pi) - e^\pi(x - \pi)^2 - \frac{e^\pi(x - \pi)^3}{3} + \frac{e^\pi(x - \pi)^5}{30} - \dots$$

which converges for all $x \in \mathbb{R}$.

Taylor Expansion Example 3

[Definition 3.33] *Taylor expansion* of $f(x) = \ln(x)$ about $a = 17$:

$$\ln(x) = \ln(17) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (x - 17)^n}{n 17^n}$$

which converges if and only if

$$0 < x \leq 34.$$

***Statistician's
Love
Taylor
Expansion***

Taylor Expansion Example 4 (from Statistics)

$X_1, \dots, X_n$ a random sample from the Normal distribution with mean μ and variance σ^2 .

$$\bar{X} = (X_1 + \dots + X_n)/n = \text{sample mean.}$$

Mean squared error of $e^{\bar{X}} = e^{2\mu}(e^{2\sigma^2/n} - 2e^{\sigma^2/(2n)} + 1)$.

Mean squared error of $e^{\bar{X}} = e^{2\mu}(e^{2\sigma^2/n} - 2e^{\sigma^2/(2n)} + 1)$.

is **EXACT** but **DIFFICULT TO INTERPRET**.

Taylor's expansion gives

$$e^{2\sigma^2/n} - 2e^{\sigma^2/(2n)} + 1 = 1 + \frac{2\sigma^2}{n} + \frac{2\sigma^4}{n^2} + \dots$$
$$-2 - \frac{\sigma^2}{n} - \frac{\sigma^4}{4n^2} + \dots + 1 \implies$$

Mean squared error of $e^{\bar{X}} = \frac{\sigma^2 e^{2\mu}}{n}$ + lower order terms

which clearly shows the effect of sample size (n).

Obtaining / Proving Taylor Expansion Convergence Regions Using 35007 Tools

- | | | |
|-------------------------|-----------------|---|
| <i>Example 4</i> | (Statistics) | Straightforward via e.g.
Definition 3.28, Theorem 3.30 |
| <i>Example 3</i> | $(\ln(x))$ | Straightforward via e.g.
Definition 3.28, Theorem 3.30 |
| <i>Example 2</i> | $(e^x \sin(x))$ | Messier but eventually
do-able via notes results |
| <i>Example 1</i> | $(1/(1 + x^2))$ | Beyond 35007
(e.g. complex analysis) |

Taylor's Theorem

The $n = 1$ version of

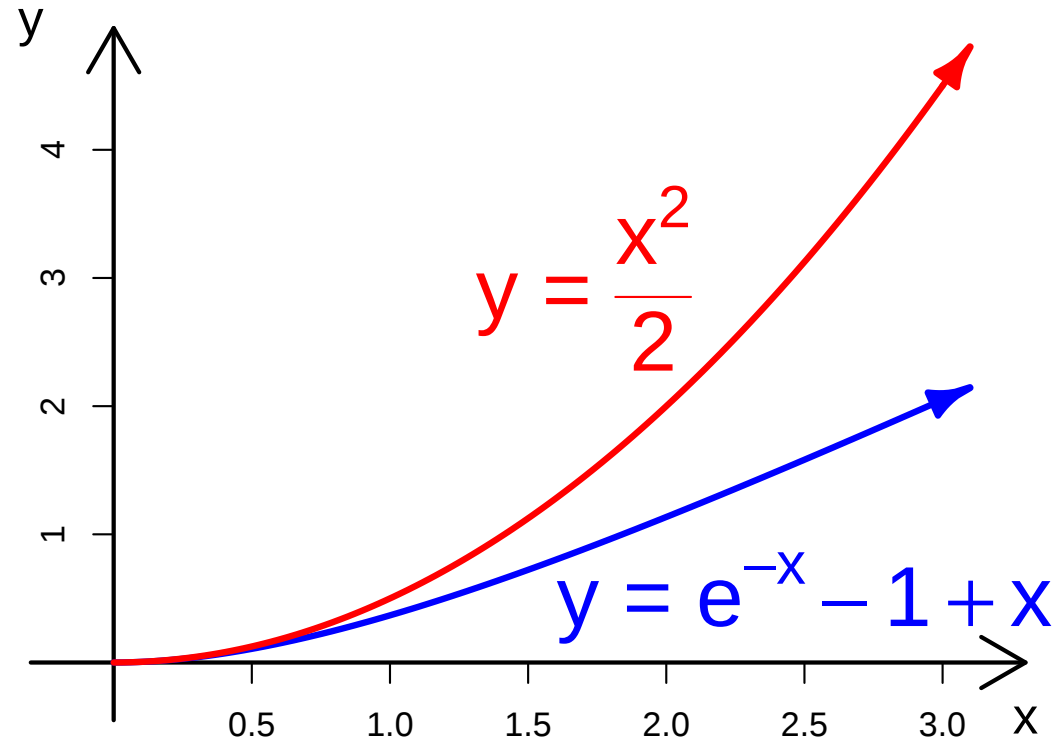
Theorem 3.34 (Taylor's Theorem):

Assuming f is “well-behaved” on an open interval containing x and a :

$$f(x) = f(a) + f'(a)(x - a) + \frac{1}{2} f''(\xi)(x - a)^2$$

for ξ between a and x .

Taylor's Theorem Example



GOAL: A formal proof that $e^{-x} - x + 1 < \frac{x^2}{2}$ for all $x > 0$.

By Taylor's Theorem (Theorem 3.34) with $a = 0$ and $n = 1$:

$$e^{-x} = 1 - x + \frac{e^{-\xi}x^2}{2} \text{ (★) for some } 0 < \xi < x.$$

Note that

$$\xi > 0 \implies -\xi < 0 \implies e^{-\xi} < e^0 = 1 \text{ (★)}.$$

Substitution of (★) into (★) gives

$$e^{-x} < 1 - x + \frac{x^2}{2} \iff e^{-x} - x + 1 < \frac{x^2}{2} \text{ for all } x > 0.$$

L'Hôpital's Rule

Our First Function Limits

1. $\lim_{x \rightarrow 3} x^2 = 9$

2. $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$

3. $\lim_{x \rightarrow \infty} \frac{\ln(x)}{\ln(x^3 + x)} = \frac{1}{3}$

$$2. \lim_{x \rightarrow 0} \frac{\sin(x)}{x}$$

$$\lim_{x \rightarrow 0} \sin(x) = \lim_{x \rightarrow 0} x = 0 \quad \longrightarrow \quad \frac{0}{0}.$$

Using L'Hôpital's Rule [[Theorem 3.15](#)]

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx} \sin(x)}{\frac{d}{dx} x} = \lim_{x \rightarrow 0} \frac{\cos(x)}{1} = 1.$$

$$3. \lim_{x \rightarrow \infty} \frac{\ln(x)}{\ln(x^3 + x)}$$

$$\lim_{x \rightarrow \infty} \ln(x) = \lim_{x \rightarrow \infty} \ln(x^3 + x) = \infty \quad \longrightarrow \quad \frac{\infty}{\infty}.$$

Using L'Hôpital's Rule [[Theorem 3.15](#)]

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\ln(x)}{\ln(x^3 + x)} &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln(x)}{\frac{d}{dx} \ln(x^3 + x)} \\ &= \lim_{x \rightarrow \infty} \frac{1/x}{(3x^2 + 1)/(x^3 + x)} \\ &= \lim_{x \rightarrow \infty} \left(\frac{x^3 + x}{3x^3 + x} \right) \\ &= \lim_{x \rightarrow \infty} \left(\frac{1 + 1/x^2}{3 + 1/x^2} \right) = \frac{1}{3}. \end{aligned}$$

Position in the Notes

***The end of these slides and the
Week 5 class***

CORRESPOND TO

the end of Chapter 3 on page 52.

The Most Relevant Exercises
(from the current subject exercise sets)

Exercise Set H

Exercise Set I
excluding I.2, I.8, I.16.

Suggestion if Time Remaining

***Halfway Point
“Town Hall”- Style
Questions and
Answers***

after a 5 minute break

