35007

Differentiation and Real Analysis

Professor Matt Wand
University of Technology Sydney

High School / Maths 1 Level

If $f(x) = 9x^6 - 4x^3 + 20x$

then what is $f'(x)$?

If $g(x) = \cos^3\left(8e^x + \sin(x+4)\sqrt{x}\right)$

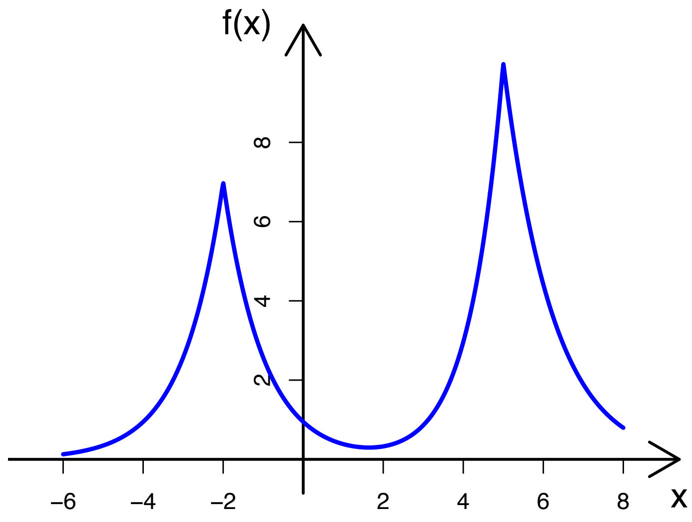
then what is $g'(x)$?

Real Analysis Level

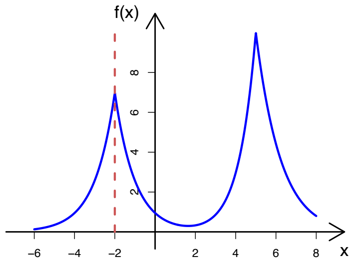
$$f(x) = 7e^{-|x+2|} + 2xe^{-|x-5|}, \; x \in \mathbb{R}.$$

1. For which $x \in \mathbb{R}$ does $f'(x)$ exist?

2. What is $f'(x)$ where it does exist?



Maths for f' existence, or not, at $x = -2$



For $h \neq 0$:

$$\frac{f(-2+h)-f(-2)}{h} = \frac{7(e^{-|h|}-1)}{h} + 2e^{-|h-7|} + \frac{4\left(e^{-7}-e^{-|h-7|}\right)}{h}.$$

Part 1: The left limit

$$\lim_{h\to 0^-}\left\{\frac{f(-2+h)-f(-2)}{h}\right\}$$

$$= \lim_{h\to 0^-}\left\{\frac{7(e^{-|h|}-1)}{h} + 2e^{-|h-7|} + \frac{4\left(e^{-7}-e^{-|h-7|}\right)}{h}\right\}.$$

Part 1: The left limit

$$\lim_{h\to 0^-}\left\{\frac{f(-2+h)-f(-2)}{h}\right\}$$

$$= \lim_{h\to 0^-}\left\{\frac{7(e^{-|h|}-1)}{h} + 2e^{-|h-7|} + \frac{4\left(e^{-7}-e^{-|h-7|}\right)}{h}\right\}.$$

For this ***left limit*** $h < 0 \implies |h| = -h \implies -|h| = h$ and $h - 7 < 0 \implies |h-7| = -(h-7) \implies -|h-7| = h - 7$ and the limit becomes (using $\lim\limits_{h\to 0}(e^h-1)/h = 1$, proven in the notes)

$$\lim_{h\to 0^-}\left\{\frac{7(e^h-1)}{h} + 2e^{h-7} - \frac{4e^{-7}\left(e^h-1\right)}{h}\right\} = 7 - 2e^{-7}.$$

Part 2: The right limit

$$\lim_{h\to 0^+}\left\{\frac{f(-2+h)-f(-2)}{h}\right\}$$

$$= \lim_{h\to 0^+}\left\{\frac{7(e^{-|h|}-1)}{h} + 2e^{-|h-7|} + \frac{4\left(e^{-7}-e^{-|h-7|}\right)}{h}\right\}.$$

Part 2: The right limit

$$\lim_{h\to 0^+}\left\{\frac{f(-2+h)-f(-2)}{h}\right\}$$

$$=\lim_{h\to 0^+}\left\{\frac{7(e^{-|h|}-1)}{h}+2e^{-|h-7|}+\frac{4\left(e^{-7}-e^{-|h-7|}\right)}{h}\right\}.$$

For this ***right limit*** $h>0 \implies |h|=h \implies -|h|=-h$ and ***assuming*** h ***below 7,*** $h-7<0 \implies |h-7|=-(h-7) \implies -|h-7|=h-7$ and the limit becomes

$$\lim_{h\to 0^+}\left\{-\frac{7(e^h-1)}{h}+2e^{h-7}-\frac{4e^{-7}\left(e^h-1\right)}{h}\right\}=-7-2e^{-7}.$$

Summary of Step 1 and Step 2

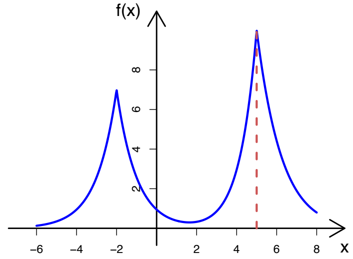
$$\lim_{h\to 0^-}\left\{\frac{f(-2+h)-f(-2)}{h}\right\}=7-2e^{-7}$$

$$\lim_{h\to 0^+}\left\{\frac{f(-2+h)-f(-2)}{h}\right\}=-7-2e^{-7}$$

WHICH ARE NOT EQUAL

$\implies f'$ does not exist at $x=-2$.

Maths for f' existence, or not, at $x=5$



For $h\neq 0$:

$$\frac{f(5+h)-f(5)}{h}=\frac{7(e^{-|7+h|}-e^{-7})}{h}+2e^{-|h|}+\frac{10\left(e^{-|h|}-1\right)}{h}.$$

After similar steps...

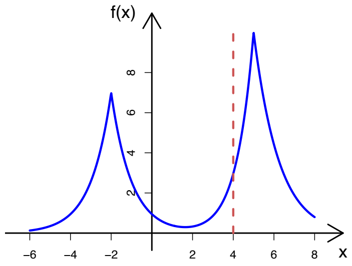
$$\lim_{h\to 0^-}\left\{\frac{f(5+h)-f(5)}{h}\right\}=12-7e^{-7}$$

$$\lim_{h\to 0^+}\left\{\frac{f(5+h)-f(5)}{h}\right\}=-8-7e^{-7}$$

WHICH ARE NOT EQUAL

$\implies f'$ does not exist at $x=5$.

Maths for f' existence, or not, at $x = 4$



For $h \neq 0$:

$$\frac{f(4+h) - f(4)}{h} = \frac{7\left(e^{-|h+6|} - e^{-6}\right)}{h} + 2e^{-|h-1|} + \frac{8\left(e^{-|h-1|} - e^{-1}\right)}{h}$$

The Full f' Result

$$f'(x) = \begin{cases} 7e^{x+2} + 2(1+x)e^{x-5}, & x < -2, \\ -7e^{-x-2} + 2(1+x)e^{x-5}, & -2 < x < 5, \\ 7e^{-x-2} + 2(1-x)e^{5-x}, & x > 5 \end{cases}$$

and f' does not exist, and not defined, at $x = -2$ and $x = 5$.

After similar steps...

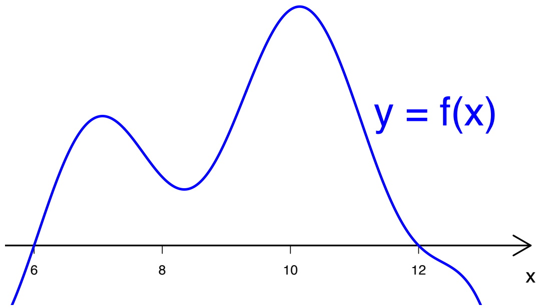
$$\lim_{h \to 0^-} \left\{ \frac{f(4+h) - f(4)}{h} \right\} = 10e^{-1} - 7e^{-6}$$

$$\lim_{h \to 0^+} \left\{ \frac{f(4+h) - f(4)}{h} \right\} = 10e^{-1} - 7e^{-6}$$

WHICH ARE EQUAL

$\implies$ f' EXISTS at $x = 4$ and $f'(4) = 10e^{-1} - 7e^{-6}$

Rolle's Theorem



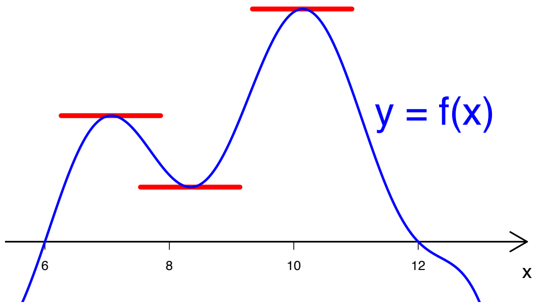
f is continuous on $[6, 12]$, differentiable on $(6, 12)$,

$f(6) = f(12) = 0 \implies f'(c) = 0$ for at least one $c \in (6, 12)$.

Example Rolle's Theorem Application

$$f(x) = 13x^5 + 3x^3 + 11x - 194, \quad x \in \mathbb{R}.$$

Prove that f has exactly one root.



f is continuous on $[6, 12]$, differentiable on $(6, 12)$,

$f(6) = f(12) = 0 \implies f'(c) = 0$ for at least one $c \in (6, 12)$.

Proof (by Contradiction)

$$f(0) = -194 \quad \text{and} \quad f(2) = 268$$

so by the Theorem 2.19 of the subject notes there is at least one

$$r \in [0, 2] \quad \text{such that } f(r) = 0.$$

$$\implies f \text{ has at least one root.}$$

The first derivative of f is

$$f'(x) = 15x^4 + 9x^2 + 11 > 0 \quad \text{for all } x \in \mathbb{R}. \ (\bigstar)$$

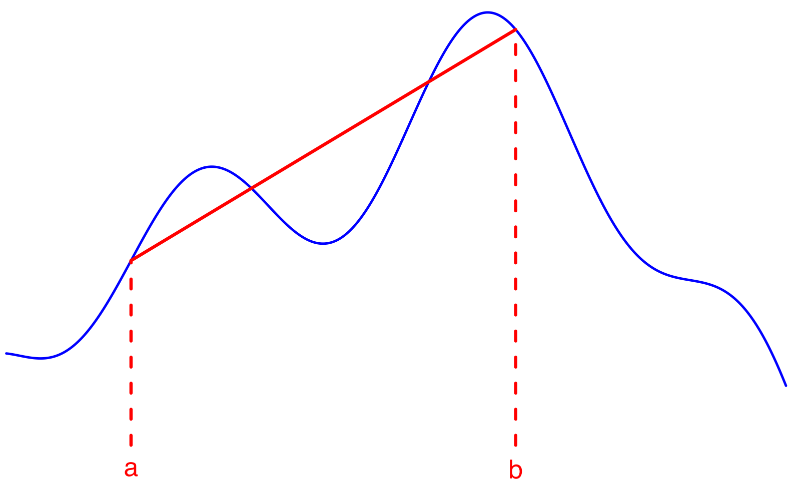
If $\widetilde{r} > r$ is another root then

$$f(r) = f(\widetilde{r}) = 0$$

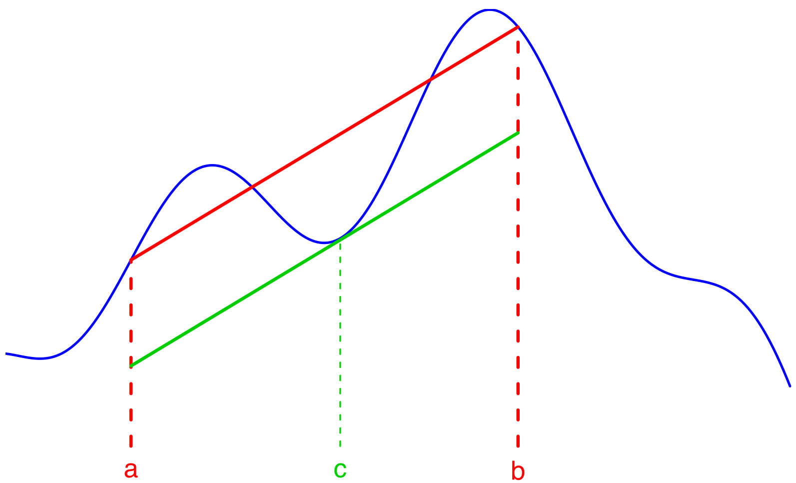
and, by Rolle's Theorem, there is $r < c < \widetilde{r}$ such that

$$f'(c) = 0 \quad \text{WHICH CONTRADICTS} \quad (\bigstar).$$

So $f(\widetilde{r}) = 0$ can't happen $\implies$ exactly one root.



The Mean Value Theorem

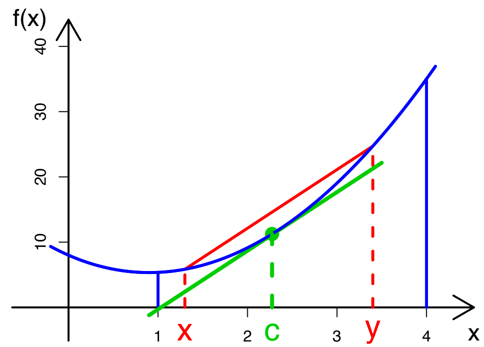


Proving Lipschitz Continuity
via the Mean Value Theorem

GOAL: Prove that $f : [1,4] \to \mathbb{R}$ given by

$$f(x) = 3x^2 - 5x + e^{-x} + 7$$

is Lipschitz continuous on $[1,4]$.



The Mean Value Theorem ensures

$$\frac{f(y)-f(x)}{y-x} = f'(c) \quad \text{for} \quad 1 \le x \le c \le y \le 4.$$

$$\implies \quad |f(y)-f(x)| \le |x-y||f'(c)| < 20|x-y| \text{ (previous slide)}$$

which proves that f is Lipschitz continuous on $[1,4]$.

Proving Lipschitz Continuity
via the Mean Value Theorem

GOAL: Prove that $f : [1,4] \to \mathbb{R}$ given by

$$f(x) = 3x^2 - 5x + e^{-x} + 7$$

is Lipschitz continuous on $[1,4]$.

Background working with Mean Value Theorem use in mind

$f'(x) = 6x - 5 - e^{-x}$; $1 \le x \le 4 \implies 1 \le 6x - 5 \le 19.$

Also $1 \le x \le 4 \implies e^{-4} \le e^{-x} \le e^{-1} \implies$

$1 + e^{-4} \le f'(x) \le 19 + e^{-1} \implies |f'(x)| < 20$ for $x \in [1,4]$.

Class Exercise

Use the Mean Value Theorem to prove that

$$f(x) = x^{31}$$

is Lipschitz continuous on $[0,1]$.

Class Exercise SOLUTION

The derivative of $f(x) = x^{31}$ is $f'(x) = 31x^{30}$. Since

$$0 \le x \le 1 \quad \Longrightarrow \quad 0 \le x^{30} \le 1 \quad \Longrightarrow \quad 0 \le 31x^{30} \le 31$$

we have $0 \le f'(x) \le 31 \implies |f'(x)| \le 31$ for $x \in [0,1]$.

Class Exercise SOLUTION

The derivative of $f(x) = x^{31}$ is $f'(x) = 31x^{30}$. Since

$$0 \le x \le 1 \quad \Longrightarrow \quad 0 \le x^{30} \le 1 \quad \Longrightarrow \quad 0 \le 31x^{30} \le 31$$

we have $0 \le f'(x) \le 31 \implies |f'(x)| \le 31$ for $x \in [0,1]$.

From the Mean Value Theorem, for $0 \le x < y \le 1$ there is a $c \in (x,y)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(c) \quad \Longrightarrow \quad \left|\frac{f(y) - f(x)}{y - x}\right| = |f'(c)|$$

$$\Longrightarrow \quad |f(x) - f(y)| = |f'(c)||x - y| \le 31|x - y|.$$

[The same argument applies for $0 \le y < x \le 1$.
The $0 \le y = x \le 1$ case is trivial.]

Class Exercise SOLUTION

The derivative of $f(x) = x^{31}$ is $f'(x) = 31x^{30}$. Since

$$0 \le x \le 1 \quad \Longrightarrow \quad 0 \le x^{30} \le 1 \quad \Longrightarrow \quad 0 \le 31x^{30} \le 31$$

we have $0 \le f'(x) \le 31 \implies |f'(x)| \le 31$ for $x \in [0,1]$.

From the Mean Value Theorem, for $0 \le x < y \le 1$ there is a $c \in (x,y)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(c) \quad \Longrightarrow \quad \left|\frac{f(y) - f(x)}{y - x}\right| = |f'(c)|$$

$$\Longrightarrow \quad |f(x) - f(y)| = |f'(c)||x - y| \le 31|x - y|.$$

The Most Relevant Exercises
(from the current subject exercise sets)

Exercise Set G

Some Admin.

Release of Assignment

The 35007 Assignment (worth 20%) is now on

matt-p-wand.net/35007.html

Release of Assignment

The 35007 Assignment (worth 20%) is now on

matt-p-wand.net/35007.html

Deadline One	Start of Class 7	$\Longrightarrow$ 1 week for Quiz 2 prep.
Deadline Two	Start of Class 8 (Quiz 2 date)	$\Longrightarrow$ assessment overlap pleas not accepted

Reminders About Week 6 (Next Week)

The Lecture / Workshop is NOT ON.

The Tutorials ARE ON.

♦ ♦ ♦