

35007

Differentiation
and
Real Analysis

Professor Matt Wand

University of Technology Sydney

High School / Maths 1 Level

If $f(x) = 9x^6 - 4x^3 + 20x$

then what is $f'(x)$?

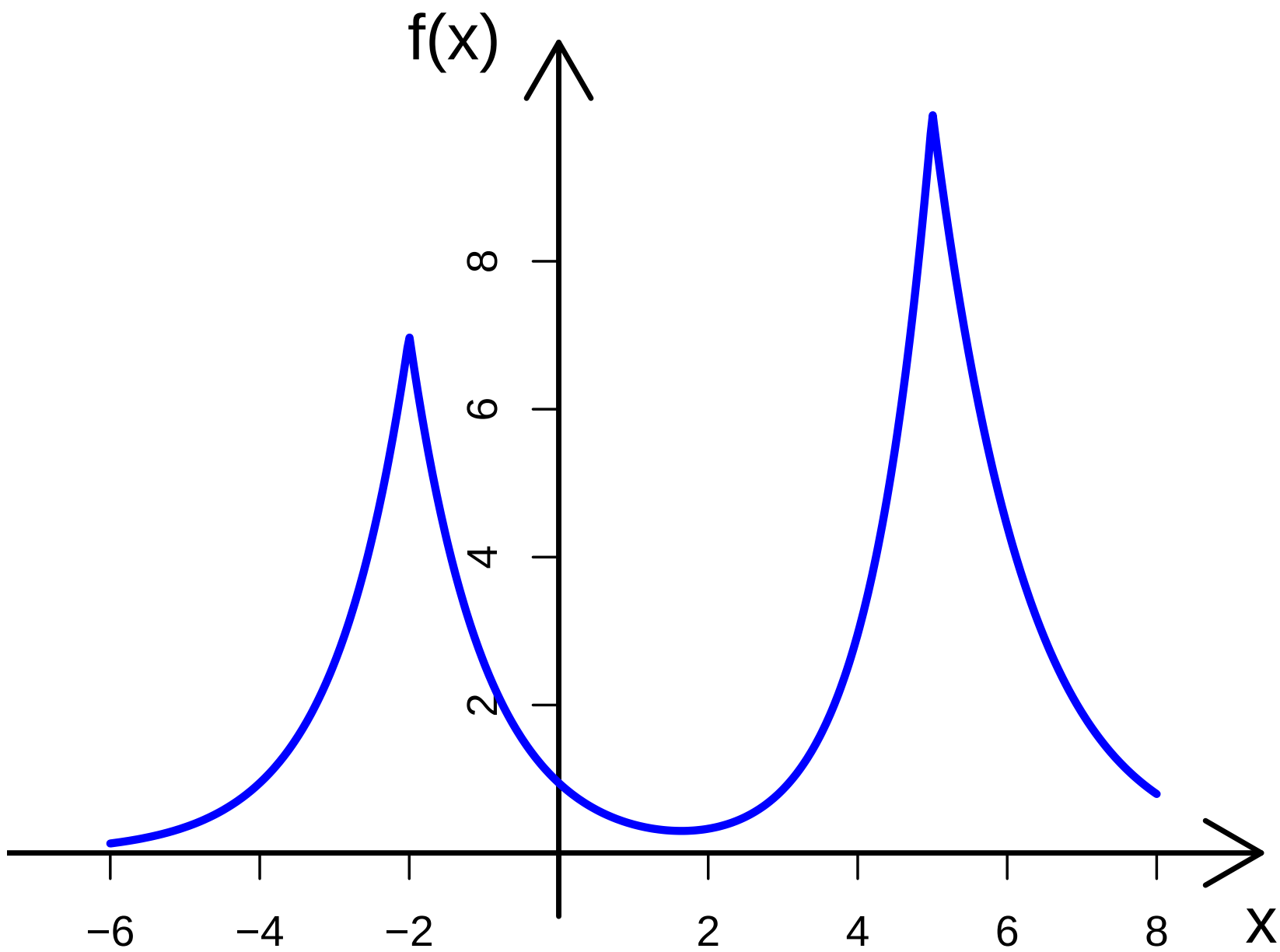
If $g(x) = \cos^3(8e^x + \sin(x + 4)\sqrt{x})$

then what is $g'(x)$?

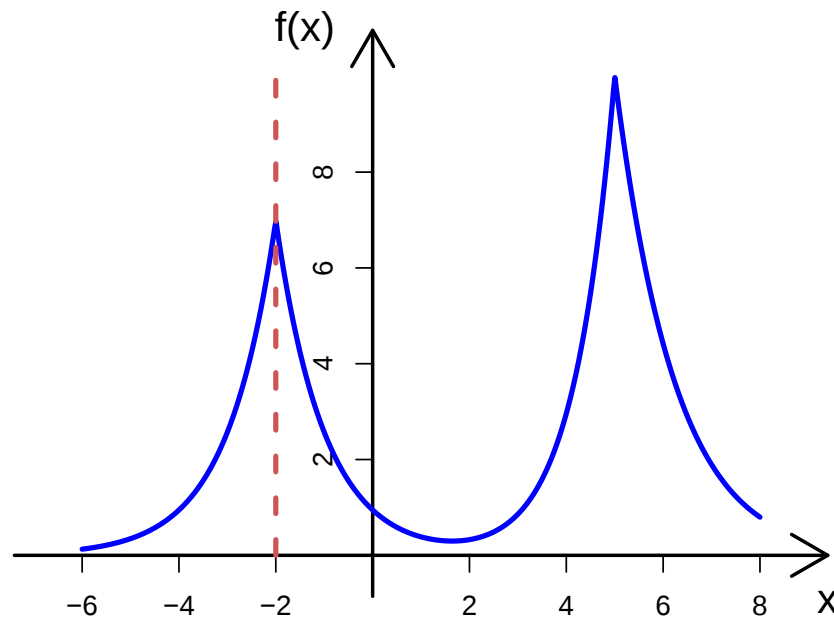
Real Analysis Level

$$f(x) = 7e^{-|x+2|} + 2xe^{-|x-5|}, \quad x \in \mathbb{R}.$$

1. For which $x \in \mathbb{R}$ does $f'(x)$ exist?
2. What is $f'(x)$ where it does exist?



Maths for f' existence, or not, at $x = -2$



For $h \neq 0$:

$$\frac{f(-2+h) - f(-2)}{h} = \frac{7(e^{-|h|} - 1)}{h} + 2e^{-|h-7|} + \frac{4(e^{-7} - e^{-|h-7|})}{h}.$$

Part 1: The left limit

$$\begin{aligned} & \lim_{h \rightarrow 0^-} \left\{ \frac{f(-2 + h) - f(-2)}{h} \right\} \\ &= \lim_{h \rightarrow 0^-} \left\{ \frac{7(e^{-|h|} - 1)}{h} + 2e^{-|h-7|} + \frac{4(e^{-7} - e^{-|h-7|})}{h} \right\}. \end{aligned}$$

Part 1: The left limit

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For this **left limit** $h < 0 \implies |h| = -h \implies -|h| = h$ and $h - 7 < 0 \implies |h - 7| = -(h - 7) \implies -|h - 7| = h - 7$ and the limit becomes (using $\lim_{h \rightarrow 0} (e^h - 1)/h = 1$, proven in the notes)

$$\lim_{h \rightarrow 0^-} \left\{ \frac{7(e^h - 1)}{h} + 2e^{h-7} - \frac{4e^{-7}(e^h - 1)}{h} \right\} = 7 - 2e^{-7}.$$

Part 2: The right limit

$$\begin{aligned} & \lim_{h \rightarrow 0^+} \left\{ \frac{f(-2 + h) - f(-2)}{h} \right\} \\ &= \lim_{h \rightarrow 0^+} \left\{ \frac{7(e^{-|h|} - 1)}{h} + 2e^{-|h-7|} + \frac{4(e^{-7} - e^{-|h-7|})}{h} \right\}. \end{aligned}$$

Part 2: The right limit

$$\lim_{h \rightarrow 0^+} \left\{ \frac{f(-2+h) - f(-2)}{h} \right\}$$
$$= \lim_{h \rightarrow 0^+} \left\{ \frac{7(e^{-|h|} - 1)}{h} + 2e^{-|h-7|} + \frac{4(e^{-7} - e^{-|h-7|})}{h} \right\}.$$

For this **right limit** $h > 0 \implies |h| = h \implies -|h| = -h$
and **assuming h below 7**, $h - 7 < 0 \implies |h - 7| = -(h - 7) \implies -|h - 7| = h - 7$ and the limit becomes

$$\lim_{h \rightarrow 0^+} \left\{ -\frac{7(e^h - 1)}{h} + 2e^{h-7} - \frac{4e^{-7}(e^h - 1)}{h} \right\} = -7 - 2e^{-7}.$$

Summary of Step 1 and Step 2

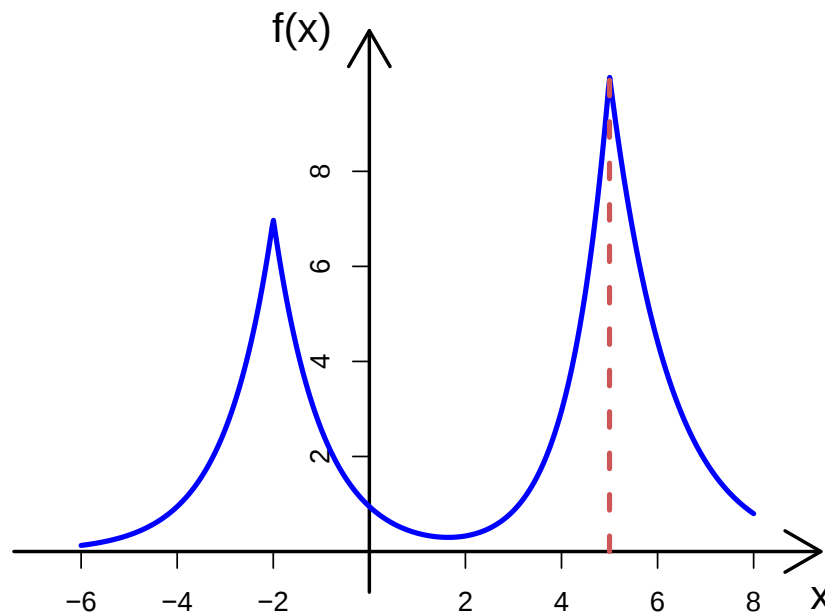
$$\lim_{h \rightarrow 0^-} \left\{ \frac{f(-2 + h) - f(-2)}{h} \right\} = 7 - 2e^{-7}$$

$$\lim_{h \rightarrow 0^+} \left\{ \frac{f(-2 + h) - f(-2)}{h} \right\} = -7 - 2e^{-7}$$

WHICH ARE NOT EQUAL

$\implies f'$ does not exist at $x = -2$.

Maths for f' existence, or not, at $x = 5$



For $h \neq 0$:

$$\frac{f(5+h) - f(5)}{h} = \frac{7(e^{-|7+h|} - e^{-7})}{h} + 2e^{-|h|} + \frac{10(e^{-|h|} - 1)}{h}.$$

After similar steps...

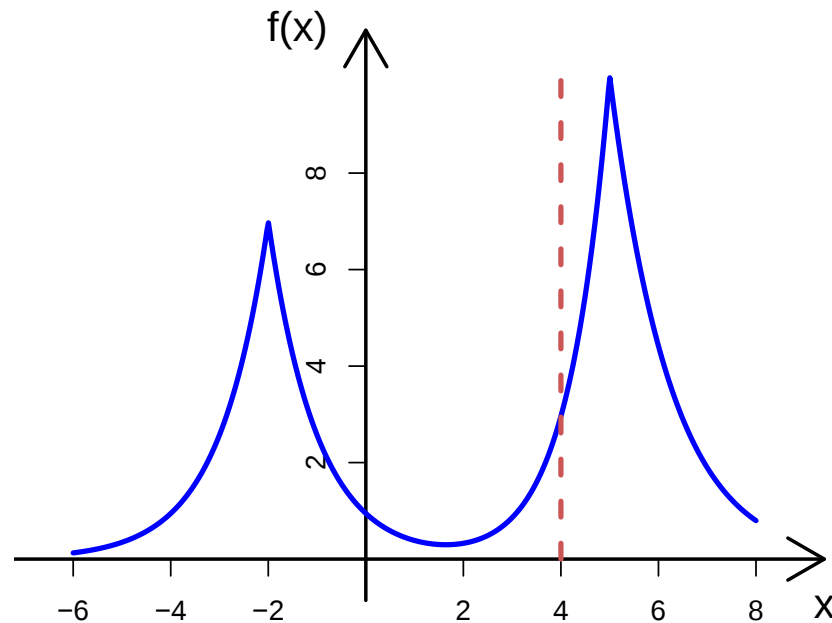
$$\lim_{h \rightarrow 0^-} \left\{ \frac{f(5+h) - f(5)}{h} \right\} = 12 - 7e^{-7}$$

$$\lim_{h \rightarrow 0^+} \left\{ \frac{f(5+h) - f(5)}{h} \right\} = -8 - 7e^{-7}$$

WHICH ARE NOT EQUAL

$\implies f'$ does not exist at $x = 5$.

Maths for f' existence, or not, at $x = 4$



For $h \neq 0$:

$$\frac{f(4+h) - f(4)}{h} = \frac{7(e^{-|h+6|} - e^{-6})}{h} + 2e^{-|h-1|} + \frac{8(e^{-|h-1|} - e^{-1})}{h}$$

After similar steps...

$$\lim_{h \rightarrow 0^-} \left\{ \frac{f(4+h) - f(4)}{h} \right\} = 10e^{-1} - 7e^{-6}$$

$$\lim_{h \rightarrow 0^+} \left\{ \frac{f(4+h) - f(4)}{h} \right\} = 10e^{-1} - 7e^{-6}$$

WHICH ARE EQUAL

$\Rightarrow f'$ **EXISTS** at $x = 4$ and

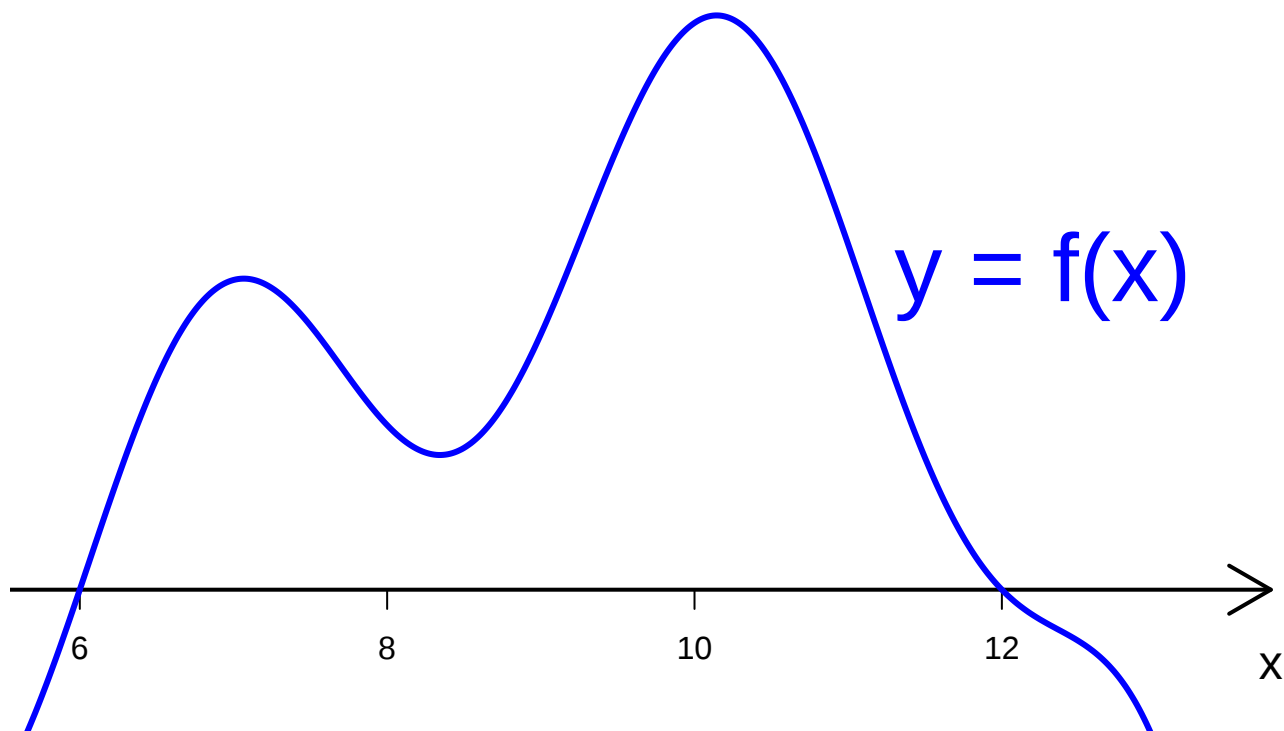
$$f'(4) = 10e^{-1} - 7e^{-6}$$

The Full f' Result

$$f'(x) = \begin{cases} 7e^{x+2} + 2(1+x)e^{x-5}, & x < -2, \\ -7e^{-x-2} + 2(1+x)e^{x-5}, & -2 < x < 5, \\ 7e^{-x-2} + 2(1-x)e^{5-x}, & x > 5 \end{cases}$$

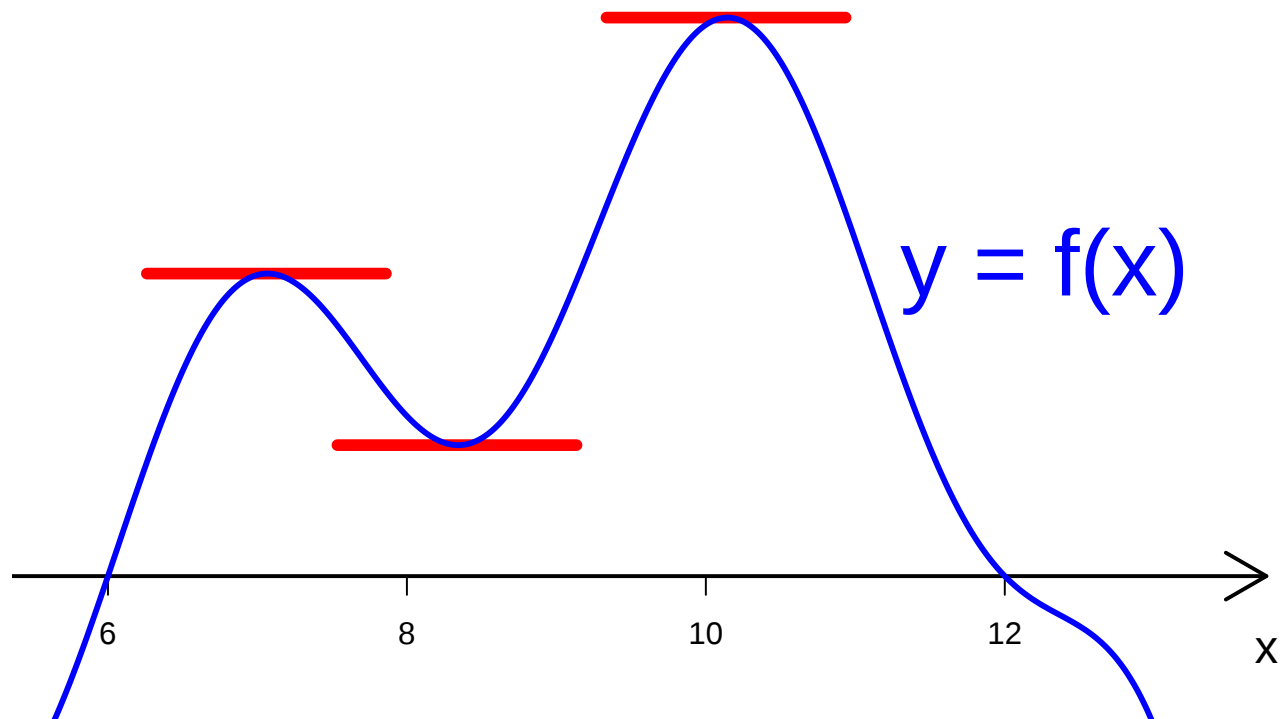
and f' does not exist, and not defined, at $x = -2$ and $x = 5$.

Rolle's Theorem



f is continuous on $[6, 12]$, differentiable on $(6, 12)$,

$f(6) = f(12) = 0 \implies f'(c) = 0$ for at least one $c \in (6, 12)$.



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Example Rolle's Theorem Application

$$f(x) = 13x^5 + 3x^3 + 11x - 194, \quad x \in \mathbb{R}.$$

Prove that f has exactly one root.

Proof (by Contradiction)

$$f(0) = -194 \quad \text{and} \quad f(2) = 268$$

so by the [Theorem 2.19](#) of the subject notes there is at least one

$$r \in [0, 2] \quad \text{such that} \quad f(r) = 0.$$

$\implies f$ has at least one root.

The first derivative of f is

$$f'(x) = 15x^4 + 9x^2 + 11 > 0 \quad \text{for all } x \in \mathbb{R}. (\star)$$

If $\tilde{r} > r$ is another root then

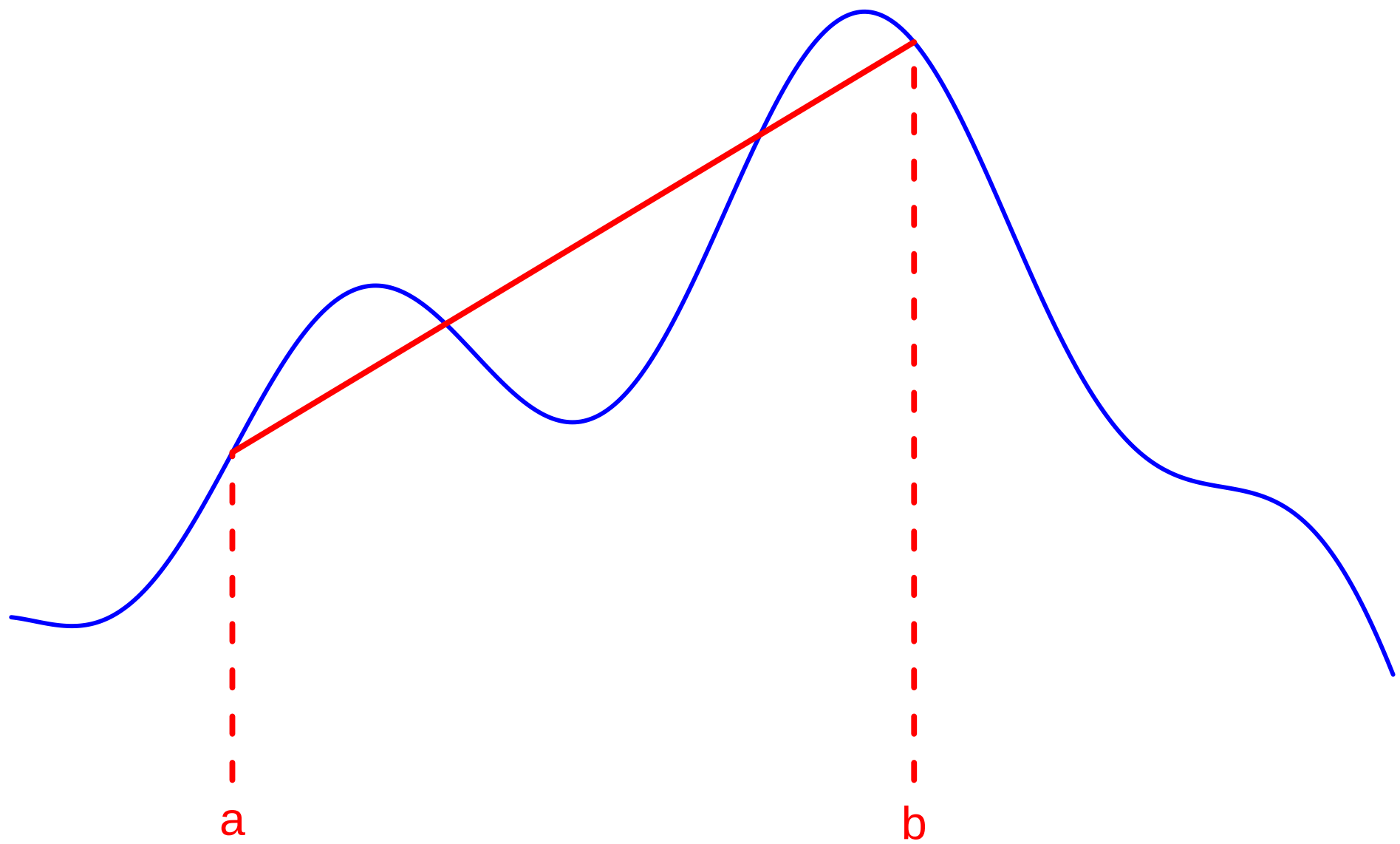
$$f(r) = f(\tilde{r}) = 0$$

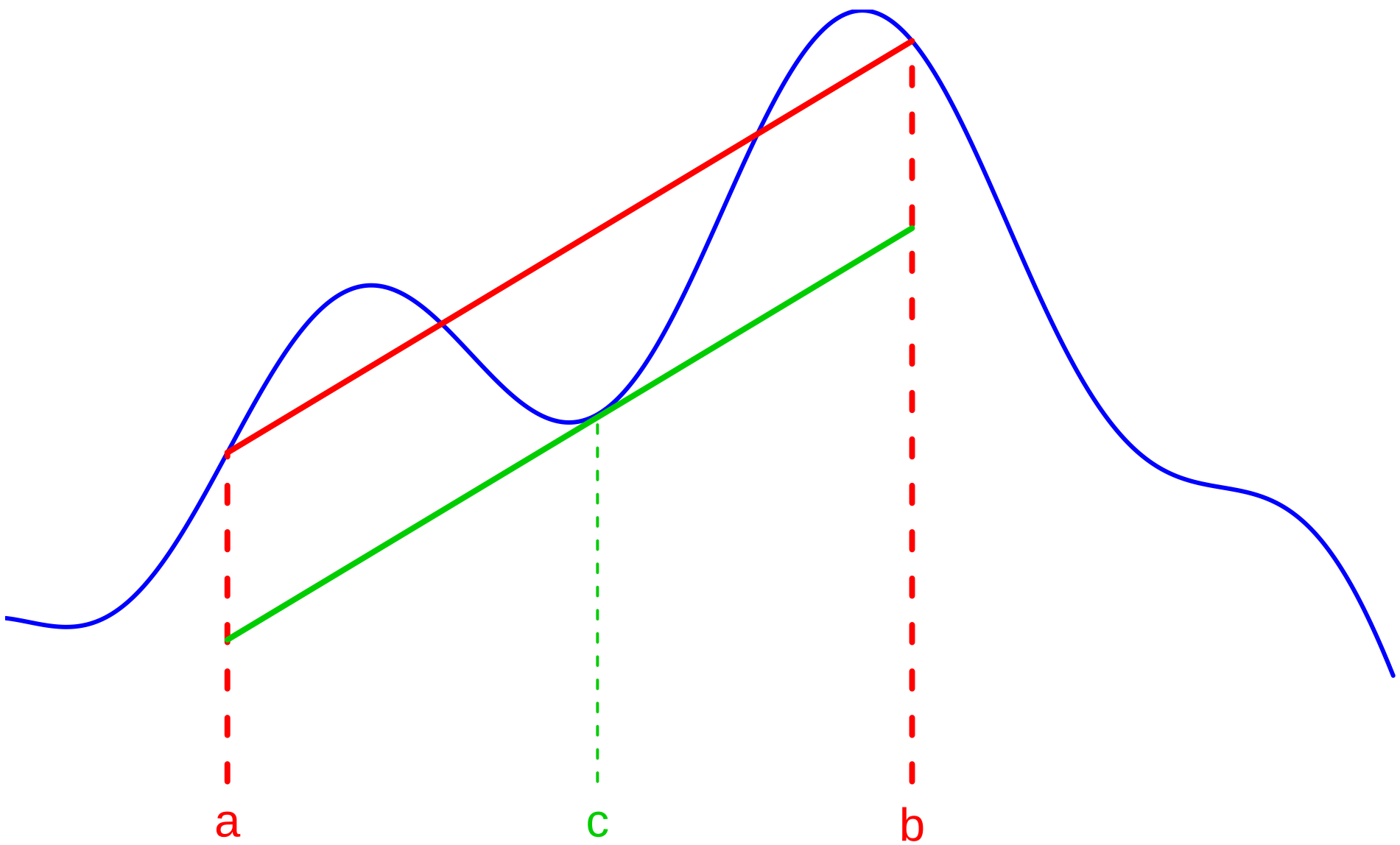
and, by **Rolle's Theorem**, there is $r < c < \tilde{r}$ such that

$$f'(c) = 0 \quad \text{WHICH CONTRADICTS} \quad (\star).$$

So $f(\tilde{r}) = 0$ can't happen $\implies$ exactly one root.

The Mean Value Theorem





Proving Lipschitz Continuity via the Mean Value Theorem

GOAL: Prove that $f : [1, 4] \rightarrow \mathbb{R}$ given by

$$f(x) = 3x^2 - 5x + e^{-x} + 7$$

is Lipschitz continuous on $[1, 4]$.

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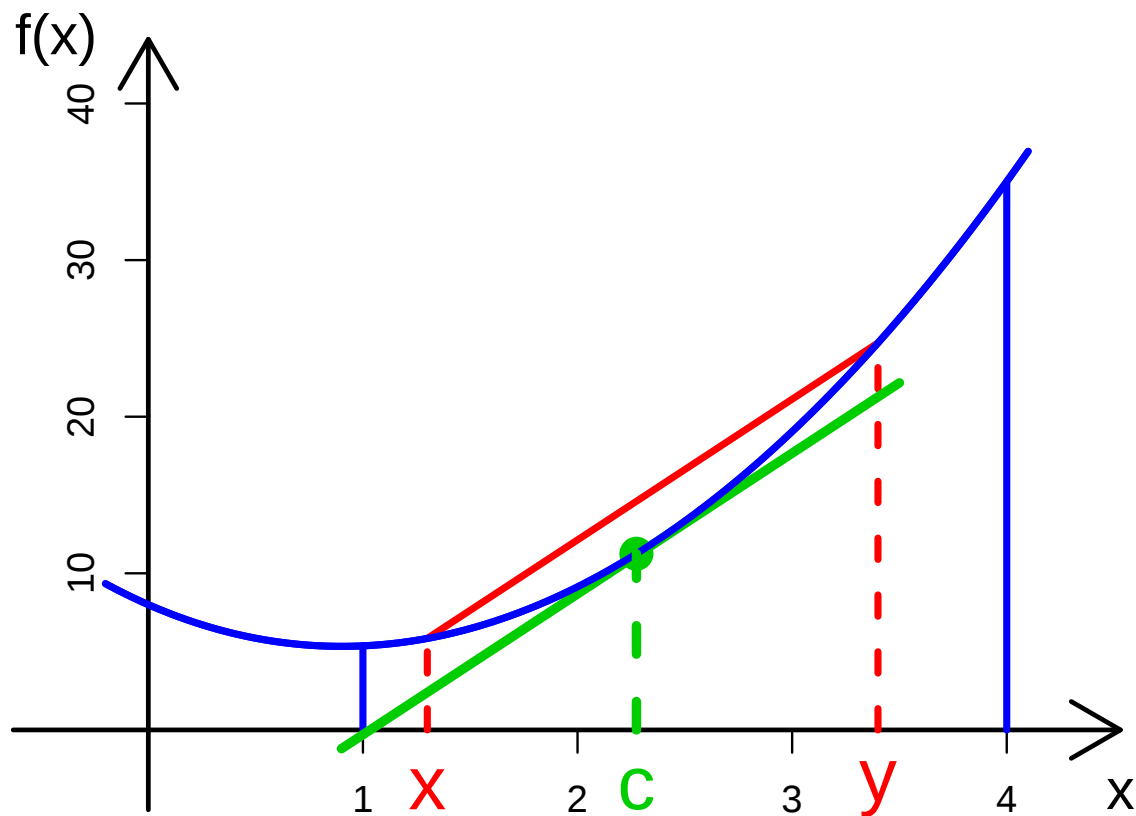
is Lipschitz continuous on $[1, 4]$.

***Background working with Mean Value Theorem
use in mind***

$$f'(x) = 6x - 5 - e^{-x} \quad ; \quad 1 \leq x \leq 4 \implies 1 \leq 6x - 5 \leq 19.$$

$$\text{Also } 1 \leq x \leq 4 \implies e^{-4} \leq e^{-x} \leq e^{-1} \implies$$

$$1 + e^{-4} \leq f'(x) \leq 19 + e^{-1} \implies |f'(x)| < 20 \text{ for } x \in [1, 4].$$



The Mean Value Theorem ensures

$$\frac{f(y) - f(x)}{y - x} = f'(c) \quad \text{for} \quad 1 \leq x \leq c \leq y \leq 4.$$

$$\implies |f(y) - f(x)| \leq |x - y| |f'(c)| < 20|x - y| \text{ (previous slide)}$$

which proves that f is Lipschitz continuous on $[1, 4]$.

Class Exercise

Use the Mean Value Theorem to prove that

$$f(x) = x^{31}$$

is Lipschitz continuous on $[0, 1]$.

Class Exercise SOLUTION

The derivative of $f(x) = x^{31}$ is $f'(x) = 31x^{30}$. Since

$$0 \leq x \leq 1 \implies 0 \leq x^{30} \leq 1 \implies 0 \leq 31x^{30} \leq 31$$

we have $0 \leq f'(x) \leq 31 \implies |f'(x)| \leq 31$ for $x \in [0, 1]$.

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From the [Mean Value Theorem](#), for $0 \leq x < y \leq 1$ there is a $c \in (x, y)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(c) \implies \left| \frac{f(y) - f(x)}{y - x} \right| = |f'(c)|$$

$$\implies |f(x) - f(y)| = |f'(c)||x - y| \leq 31|x - y|.$$

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$$\implies |f(x) - f(y)| = |f'(c)||x - y| \leq 31|x - y|.$$

[The same argument applies for $0 \leq y < x \leq 1$.

The $0 \leq y = x \leq 1$ case is trivial.]

The Most Relevant Exercises
(from the current subject exercise sets)

Exercise Set G

*Some
Admin.*

Release of Assignment

***The 35007 Assignment
(worth 20%) is now on
matt-p-wand.net/35007.html***

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Deadline One	Start of Class 7	⇒ 1 week for Quiz 2 prep.
Deadline Two	Start of Class 8 (Quiz 2 date)	⇒ assessment overlap pleas not accepted

Reminders About Week 6 (Next Week)

*The Lecture / Workshop is **NOT ON**.*

*The Tutorials **ARE ON**.*

