UNIVERSITY OF TECHNOLOGY SYDNEY
School of Mathematical and Physical Sciences
35007 Real Analysis

Supplementary Results

The subject's main notes are

AN INTRODUCTION TO REAL ANALYSIS by MARK CRADDOCK.

This document contains a few additional results that are useful in various contexts, such as solving some of the subject's exercises. The proofs of the results are deferred to an appendix.

Date of this version: **22nd July 2026**[1]

[1]This document is less than a few months old which makes errors more likely. Updated versions with later dates may be released during the semester. Notifications will be sent via e-mail whenever an updated version is released.

Notational Alternatives Regarding Sequence Limits. Starting with Definition 1.16 on page 8, the subject's main notes use the notation

$$x_n \to x$$

to denote the sequence $\{x_n\}_{n=1}^{\infty}$ converging to the real number x. A commonly used alternative notation is

$$\lim_{n\to\infty} x_n = x.$$

These supplementary results and many of the exercises and solutions use the latter notation.

THEOREM S.1 *(The Ratio Test for Sequences) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence such that $x_n \neq 0$ for all $n \in \mathbb{N}$ and*

$$L = \lim_{n\to\infty} \left| \frac{x_{n+1}}{x_n} \right| \quad \textit{exists.}$$

If $L < 1$ then $\lim_{n\to\infty} x_n = 0$. If $L > 1$ then x_n diverges. Also, if L does not exist and

$$\liminf_{n\to\infty} \left| \frac{x_{n+1}}{x_n} \right| = \infty \quad \textit{then } x_n \textit{ diverges.}$$

THEOREM S.2 *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions on a set $X \subseteq \mathbb{R}$. Then f_n converges uniformly to the function f on X if and only if*

$$\lim_{n\to\infty} \sup \big\{ |f_n(x) - f(x)| : x \in X \big\} = 0.$$

Appendix: Proofs

Proof of Theorem S.1

We treat the following three cases separately:

(i) $L < 1$, (ii) $L > 1$ and (iii) L does not exist and $\liminf_{n\to\infty} \big|x_{n+1}/x_n\big| = \infty$.

First suppose that $L < 1$. Then $(1-L)/2 > 0$ so there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\big||x_{n+1}|/|x_n| - L\big| < \tfrac{1}{2}(1-L) \quad \Longleftrightarrow \quad -\tfrac{1}{2}(1-L)|x_n| < |x_{n+1}| - L|x_n| < \tfrac{1}{2}(1-L)|x_n|$$

which implies that

$$|x_{n+1}| < \overline{L}\,|x_n| \quad \text{for all} \quad n \geq N \quad \text{where} \quad \overline{L} = \tfrac{1}{2}(L+1). \tag{1}$$

From (1) with $n = N$ we have

$$|x_{N+1}| < \overline{L}|x_N|. \tag{2}$$

Then from (1) with $n = N+1$ and then application of (2) we have

$$|x_{N+2}| < \overline{L}\,|x_{N+1}| < \overline{L}^2|x_N|.$$

Setting $n = N+2$ in (1) and another application of (2) leads to

$$|x_{N+3}| < \overline{L}\,|x_{N+2}| < \overline{L}^2\,|x_{N+1}| < \overline{L}^3|x_N|.$$

Continuing in this fashion we obtain the result

$$|x_{N+m}| < |x_N|\overline{L}^m \quad \text{for all} \quad m \in \mathbb{N}. \tag{3}$$

The substitution of $n = N+m$ into (3) leads to

$$|x_n| < \big(|x_N|/\overline{L}^N\big)\overline{L}^n \quad \text{for all} \quad n \geq N+1.$$

After some simple algebra we then have

$$0 < |x_n| < C_N \exp(-n\kappa) \quad \text{for all} \quad n \geq N+1$$

where $C_N \equiv |x_N|\Big/\overline{L}^N$ and $\kappa = -\ln(\overline{L})$. Since

$$L < 1 \quad \Longrightarrow \quad \tfrac{1}{2}L < \tfrac{1}{2} \quad \Longrightarrow \quad \tfrac{1}{2}L + \tfrac{1}{2} < 1$$

we have

$$\overline{L} < 1 \quad \Longrightarrow \quad \ln(\overline{L}) < 0 \quad \Longrightarrow \quad \kappa > 0.$$

Hence

$$\lim_{n\to\infty} \big\{C_N \exp(-n\kappa)\big\} = 0.$$

By the Squeeze Theorem (Theorem 2.8)

$$\lim_{n\to\infty} |x_n| = 0 \qquad \Longrightarrow \qquad \lim_{n\to\infty} x_n = 0.$$

Now suppose that $L > 1$ and let $\widetilde{x}_n = 1/x_n$. Then using Theorem 2.8 and the fact that the function $g(x) = 1/x$ is continuous on $(0, \infty)$ have

$$\lim_{n\to\infty} \left| \frac{\widetilde{x}_{n+1}}{x_n} \right| = \lim_{n\to\infty} \left| \frac{1}{x_{n+1}/x_n} \right| = \frac{1}{\lim\limits_{n\to\infty} |x_{n+1}/x_n|} = \frac{1}{L} < 1.$$

Therefore, from the first (just proven) part of the theorem

$$\lim_{n\to\infty} \widetilde{x}_n = 0 \quad \text{and so} \quad x_n = \frac{1}{\widetilde{x}_n} \quad \text{diverges}.$$

Now consider the case where L does not exist but $\liminf\limits_{n\to\infty} |x_{n+1}/x_n| = \infty$. Once again, let $\widetilde{x}_n = 1/x_n$. Then

$$\limsup_{n\to\infty} \left| \frac{\widetilde{x}_{n+1}}{x_n} \right| = \limsup_{n\to\infty} \left| \frac{1}{x_{n+1}/x_n} \right| \le \frac{1}{\liminf\limits_{n\to\infty} |x_{n+1}/x_n|} = 0 < 1.$$

Therefore,

$$\lim_{n\to\infty} \left| \frac{\widetilde{x}_{n+1}}{x_n} \right| = 0$$

and from the first part of the theorem

$$\lim_{n\to\infty} \widetilde{x}_n = 0 \quad \text{which implies that} \quad x_n = \frac{1}{\widetilde{x}_n} \quad \text{diverges}.$$

Proof of Theorem S.2

Suppose that f_n converges uniformly to the function f on X. Then by Definition 5.2, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \ge N$ implies

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } x \in X. \tag{4}$$

Now note that (4) implies that

$$\sup \big\{ |f_n(x) - f(x)| : x \in X \big\} < \varepsilon. \tag{5}$$

If $a_n = \sup \big\{ |f_n(x) - f(x)| : x \in X \big\}$ then (5) is equivalent to

$$|a_n - 0| < \varepsilon$$

and by Definition 1.16 we have

$$\lim_{n\to\infty} a_n = 0.$$

Therefore, if f_n converges to f uniformly on X then

$$\lim_{n\to\infty} \sup \big\{ |f_n(x) - f(x)| : x \in X \big\} = 0. \tag{6}$$

To prove the converse, suppose that (6) holds. Then from Definition 1.16, for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \ge N$ implies

$$\sup \big\{ |f_n(x) - f(x)| : x \in X \big\} < \varepsilon \qquad \Longrightarrow \qquad |f_n(x) - f(x)| < \varepsilon \quad \text{for all } x \in X.$$

So by Definition 5.2, f_n converges uniformly to f on X.