UNIVERSITY OF TECHNOLOGY SYDNEY

School of Mathematical and Physical Sciences

35007 Real Analysis

Solutions to the Quiz 1 Practice Exercises

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A.5. *Relevant page(s) in the notes:* 6

The set of interest is $A = \{19, -14, 33, 0, 9, -11\}$ which has a finite number of elements. Clearly

$$-14 = \text{lowest element of } A$$

is a lower bound on A. Also, all other lower bounds on A are less than -14. Therefore

$$-14 = \text{the greatest lower bound on } A = \inf A.$$

Also,

$$33 = \text{highest element of } A$$

is an upper bound on A. Also, all other upper bounds on A are greater than 33. Therefore

$$33 = \text{the least upper bound on } A = \sup A.$$

In summary,

$$\inf A = -14 \quad \text{and} \quad \sup A = 33.$$

A.6. *Relevant page(s) in the notes:* 6

As is well-known, the five lowest prime numbers are

$$2, 3, 5, 7 \text{ and } 11.$$

All other prime numbers are higher, so 2 is a lower bound on $\mathbb{P}$. All other lower bounds on $\mathbb{P}$ are less than 2 and so

$$2 = \text{the greatest lower bound on } \mathbb{P} = \inf \mathbb{P}.$$

It is well-known from number theory that there is an infinite number of prime numbers. Therefore, for any real number $x \geq 2$ there exists $p \in \mathbb{P}$, such that $p \geq x$. This implies that no finite real number can be an upper bound on $\mathbb{P}$ and so

$$\infty = \text{the least upper bound on } \mathbb{P} = \sup \mathbb{P}.$$

In summary

$$\inf \mathbb{P} = 2 \quad \text{and} \quad \sup \mathbb{P} = \infty.$$

[1] This document is less than a few months old which makes errors more likely. Updated versions with later dates may be released during the semester. Notifications will be sent via e-mail whenever an updated version is released.

A.7. $\boxed{\textit{Relevant page(s) in the notes: } 6}$

The set

$$\{m/n : m, n \in \mathbb{N}, m + n \leq 10\}.$$

has a finite number of elements consisting of all unique fractions of integers between 1 and 10 inclusive subject to the constraint that the sum of the numerator and denominator must not exceed 10. Some of the elements are

$$7/4, \quad 2/9, \quad 5/1 \quad \text{and} \quad 5/8.$$

If enumerated and equalities such $1/2 = 2/4 = 3/6 = 4/8 = 5/10$ are taken into account then the set has 31 unique entries. Based on these considerations, it is apparent that the lowest possible fraction in the set is $1/9$ and the highest one is $9/1 = 9$. Therefore

$$\inf\{m/n : m, n \in \mathbb{N}, m + n \leq 10\} = 1/9 \quad \text{and} \quad \sup\{m/n : m, n \in \mathbb{N}, m + n \leq 10\} = 9.$$

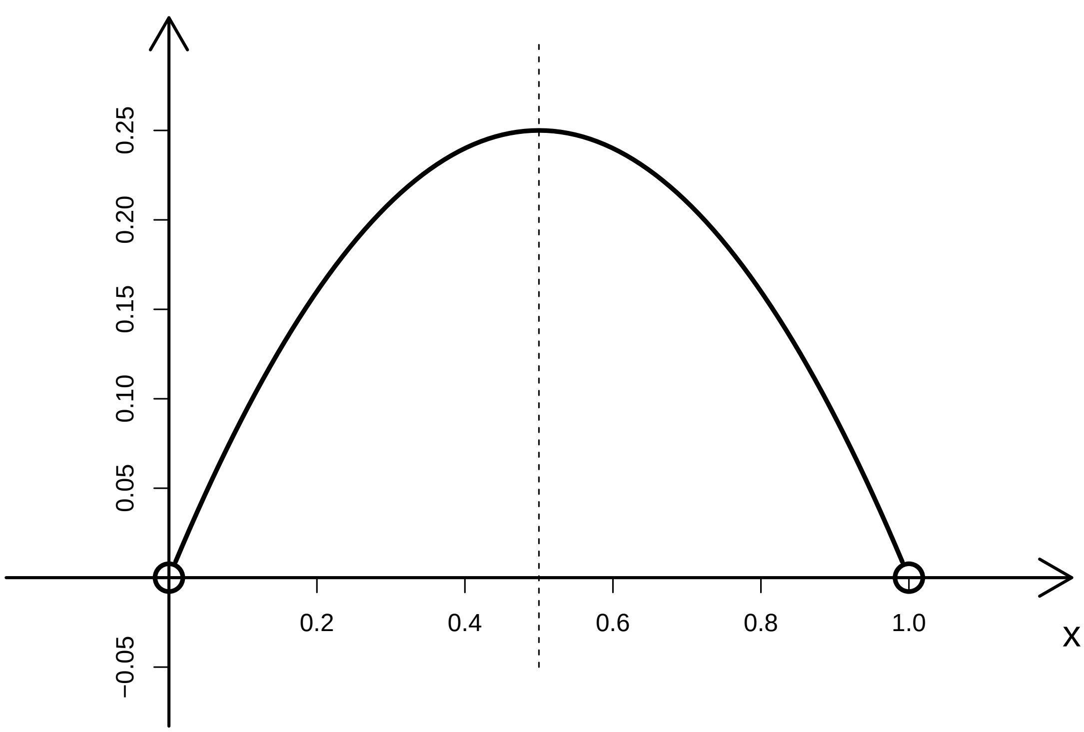
A.8. $\boxed{\textit{Relevant page(s) in the notes: } 6}$

The theory of quadratic functions has established that a function of the form ax^2+bx+c such that $a < 0$ is an arch-shaped (as opposed to bowl-shaped) parabola with axis of symmetry at $x = -b/(2a)$. Therefore the function $x - x^2$ is an arch-shaped parabola with axis of symmetry at $x = \frac{1}{2}$. This implies that the parabola's vertex, and maximum, is at the vertical position

$$\tfrac{1}{2} - (\tfrac{1}{2})^2 = \tfrac{1}{4}.$$

The accompanying figure provides illustration. Since this maximum occurs inside the interval $(0, 1)$ $\frac{1}{4}$ is the maximum value of the set in the exercise. Hence $\frac{1}{4}$ is an upper bound for $\{x - x^2 : 0 < x < 1\}$ and the lowest number that has this property. Thus

$$\sup\{x - x^2 : 0 < x < 1\} = \tfrac{1}{4}.$$



Regarding the infimum, it is straightforward to show and apparent from the accompanying figure that 0 is a lower bound for $\{x - x^2 : 0 < x < 1\}$ and is greater than all negative real numbers (which are also lower bounds). Any $0 < \varepsilon \leq \frac{1}{4}$ is not a lower bound for $\{x - x^2 : 0 < x < 1\}$ since

$$(\varepsilon/2) - (\varepsilon/2)^2 \in \{x - x^2 : 0 < x < 1\} \quad \text{but} \quad \varepsilon > (\varepsilon/2) > (\varepsilon/2) - (\varepsilon/2)^2.$$

Hence

$$\inf\{x - x^2 : 0 < x < 1\} = 0.$$

A.9. *Relevant page(s) in the notes:* 6

This is publication-quality proof. See † at the end of this document for a full explanation.

Let $A = \{n/(n+1) : n \in \mathbb{N}\}$ be the set of interest. Noting that

$$A = \{1/2, 2/3, 3/4, \ldots\}$$

we see that

$$\frac{1}{2} \leq n/(n+1) < 1 \quad \text{for all } n \in \mathbb{N}. \tag{1}$$

Therefore $1/2$ is lower bound on A and greater than any other lower bound (e.g. $1/4$, -7). Hence

$$\inf\{n/(n+1) : n \in \mathbb{N}\} = 1/2.$$

From (1) it is clear that 1 is an upper bound on the set in the exercise. For any $0 < \varepsilon < 1$ let $n_\varepsilon \in \mathbb{N}$ be such that

$$n_\varepsilon > \varepsilon^{-1} - 1.$$

Then

$$1/n_\varepsilon < 1/(\varepsilon^{-1} - 1) \quad \Longrightarrow \quad 1 + 1/n_\varepsilon < 1 + 1/(\varepsilon^{-1} - 1) = 1/(1 - \varepsilon).$$

Taking reciprocals on both sides of the last inequality we get

$$1/(1 + 1/n_\varepsilon) > 1 - \varepsilon \quad \Longleftrightarrow \quad n_\varepsilon/(n_\varepsilon + 1) > 1 - \varepsilon.$$

Therefore, $1 - \varepsilon$ is not an upper bound for A for any $0 < \varepsilon < 1$. This means that 1 is the lowest, or least, upper bound for A. Hence

$$\sup\{n/(n+1) : n \in \mathbb{N}\} = 1.$$

Preliminary background working on how an appropriate n_ε was obtained

For a given $0 < \varepsilon < 1$, we need find n big enough so that

$$\frac{n}{n+1} > 1 - \varepsilon.$$

Taking reciprocals of both sides gives

$$1 + \frac{1}{n} < \frac{1}{1 - \varepsilon} \quad \Longleftrightarrow \quad \frac{1}{n} < \frac{1}{1 - \varepsilon} - 1 = \frac{\varepsilon}{1 - \varepsilon}$$

Taking reciprocals again gives

$$n > \frac{1 - \varepsilon}{\varepsilon} = \varepsilon^{-1} - 1.$$

So taking $n_\varepsilon \in \mathbb{N}$ to be bigger than $\varepsilon^{-1} - 1$ will get us what we need.

B.1. $\boxed{\textit{Relevant page(s) in the notes: } 8}$

We treat the following two cases separately:

(i) $c = 0$ and (ii) $c \neq 0$.

First suppose that $c = 0$. Then for all $\varepsilon > 0$

$$n \geq 1 \quad \Longrightarrow \quad |ca_n - ca| = |0 - 0| = 0 < \varepsilon.$$

Therefore, by Definition 1.16,

$$\lim_{n\to\infty} (ca_n) = ca \quad \text{when} \quad c = 0.$$

Now suppose that $c \neq 0$. Since

$$\lim_{n\to\infty} a_n = a \tag{2}$$

for any $\varepsilon > 0$ it is also true that $\varepsilon/|c| > 0$ and so, by (2), there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \Longrightarrow \quad |a_n - a| < \varepsilon/|c|.$$

Hence,

$$n \geq N \quad \Longrightarrow \quad |ca_n - ca| = |c||a_n - n| < |c|(\varepsilon/|c|) = \varepsilon$$

and so by Definition 1.16

$$\lim_{n\to\infty} (ca_n) = ca \quad \text{when} \quad c \neq 0.$$

B.2. $\boxed{\textit{Relevant page(s) in the notes: } 8}$

$\boxed{\textit{This is publication-quality proof. See † at the end of this document for a full explanation.}}$

For any given $\varepsilon > 0$ set $N = \lceil 7/\varepsilon \rceil$. Then $n \geq N$ implies that

$$n \geq 7/\varepsilon \implies n\varepsilon \geq 7 \implies n\varepsilon + 2\varepsilon > 7 \implies 7/(n+2) < \varepsilon.$$

Noting that $7/(n+2) = |3(n-1)/(n+2) - 3|$ for all $n \in \mathbb{N}$ we then have

$$n \geq N \implies \left| \frac{3n-1}{n+2} - 3 \right| < \varepsilon.$$

Therefore, by Definition 1.16,

$$\lim_{n\to\infty} \frac{3n-1}{n+2} = 3.$$

Preliminary background working that led to an effective N choice

For each $\varepsilon > 0$ need to get to

$$\left| \frac{3n-1}{n+2} - 3 \right| < \varepsilon. \tag{3}$$

Simplification of left-hand side of (3) leads to the equivalent condition

$$\frac{7}{n+2} < \varepsilon \quad \Longleftrightarrow \quad n > 7/\varepsilon - 2$$

The simpler condition $n \geq 7/\varepsilon$ implies $n > 7/\varepsilon - 2$ and avoids having to deal with the possibility of negative numbers on the right-hand side (e.g. if $\varepsilon = 10$). So setting

$$N = \lceil 7/\varepsilon \rceil = \textit{the smallest integer greater than or equal to } 7/\varepsilon$$

will allow us to say that $n \geq N$ implies (3).

B.3. $\boxed{\textit{Relevant page(s) in the notes: 6, 8}}$

$\boxed{\textit{This is publication-quality proof. See † at the end of this document for a full explanation.}}$

For any given $\varepsilon > 0$ set

$$N = \left\lceil \sqrt{24/(49\varepsilon)} \right\rceil.$$

Then $n \geq N$ implies that, with use of the triangle inequality,

$$n \geq \sqrt{24/(49\varepsilon)} \implies 49\varepsilon n^2 \geq 24 \implies 49\varepsilon n^3 \geq 24n \geq 21 + 3n \geq |21 - 3n|.$$

However, for all $n \in \mathbb{N}$,

$$49\varepsilon n^3 \geq |21 - 3n| \quad \implies \quad 49\varepsilon n^3 + 21n\varepsilon > |21 - 3n| \quad \iff \quad \left| \frac{21 - 3n}{49\varepsilon n^3 + 21n} \right| < \varepsilon.$$

Noting that

$$\frac{21 - 3n}{49n^3 + 21n} = \frac{n^3 + 3}{7n^3 + 3n} - \frac{1}{7} \quad \text{for all } n \in \mathbb{N}$$

we then have

$$n \geq N \implies \left| \frac{n^3 + 3}{7n^3 + 3n} - \frac{1}{7} \right| < \varepsilon.$$

Therefore, by Definition 1.16,

$$\lim_{n \to \infty} \frac{n^3 + 3}{7n^3 + 3n} = \frac{1}{7}.$$

<u>*Preliminary background working that led to an effective N choice*</u>

For each $\varepsilon > 0$ need to get to

$$\left| \frac{n^3 + 3}{7n^3 + 3n} - \frac{1}{7} \right| < \varepsilon \tag{4}$$

Some simple algebraic steps lead to

$$\frac{n^3 + 3}{7n^3 + 3n} - \frac{1}{7} = \frac{|21 - 3n|}{49n^3 + 21n}.$$

By the triangle inequality, for all $n \in \mathbb{N}$,

$$|21 - 3n| \leq 21 + 3n \leq 21n + 3n = 24n$$

Also, for all $n \in \mathbb{N}$,

$$49n^3 + 21n > 49n^3 \quad \implies \quad \frac{1}{49n^3 + 21n} < \frac{1}{49n^3}.$$

On combining these last three equations

$$\left|\frac{n^3+3}{7n^3+3n}-\frac{1}{7}\right|<\frac{24n}{49n^3}=\frac{24}{49n^2}.$$

Therefore having n big enough so that $24/(49n^2)\le\varepsilon$ implies (4) for such n. Therefore, the choice $N=\left\lceil\sqrt{24/(49\varepsilon)}\right\rceil$ is such that $n\ge N$ leads to what is needed to arrive at (4).

B.4. *Relevant page(s) in the notes:* 8

We treat the following three cases separately:

(i) $a=0$, (ii) $a>0$ and (iii) $a<0$.

First suppose that $a=0$. Since

$$\lim_{n\to\infty}a_n=0 \tag{5}$$

for any $\varepsilon>0$ it is also true that $\varepsilon^2>0$ and so, by (5), there exists $N\in\mathbb{N}$ such that

$$n\ge N \implies |a_n|<\varepsilon^2 \implies \sqrt{|a_n|}<\varepsilon \iff \left|\sqrt{a_n}-\sqrt{0}\right|<\varepsilon. \tag{6}$$

Therefore, by Definition 1.16

$$\lim_{n\to\infty}\sqrt{a_n}=\sqrt{a} \quad \text{when} \quad a=0.$$

Now suppose that $a>0$. Since

$$\lim_{n\to\infty}a_n=a \tag{7}$$

for any $\varepsilon>0$ it is also true that $\varepsilon\sqrt{a}>0$ and so, by (7), there exists $N\in\mathbb{N}$ such that

$$n\ge N \implies |a_n-a|<\varepsilon\sqrt{a}. \tag{8}$$

Next note that

$$a_n-a=(\sqrt{a_n})^2-(\sqrt{a})^2=(\sqrt{a_n}-\sqrt{a})(\sqrt{a_n}+\sqrt{a}) \implies \sqrt{a_n}-\sqrt{a}=\frac{a_n-a}{\sqrt{a_n}+\sqrt{a}}. \tag{9}$$

Hence, using (8) and (9),

$$n\ge N \implies \left|\sqrt{a_n}-\sqrt{a}\right|=\frac{|a_n-a|}{\sqrt{a_n}+\sqrt{a}}\le\frac{|a_n-a|}{\sqrt{a}}<\frac{\varepsilon\sqrt{a}}{\sqrt{a}}=\varepsilon.$$

Therefore, by Definition 1.16

$$\lim_{n\to\infty}\sqrt{a_n}=\sqrt{a} \quad \text{when} \quad a>0.$$

Finally, we show that $a<0$ is not possible when (7) holds with all $a_n\ge 0$. We do this by contradiction. If $a<0$ then $\varepsilon=-a/2>0$. Then, from (7) there is $N\in\mathbb{N}$ such that

$$n\ge N \implies |a_n-a|<-a/2 \iff a/2<a_n-a<-a/2 \implies a_n<a/2<0.$$

But this $a_n<0$ result contradicts the $a_n\ge 0$ assumption. Therefore, $a<0$ is not possible and this case can be excluded.

B.5. $\boxed{\textit{Relevant page(s) in the notes: } 7, 8}$

We will use proof by contradiction. Suppose that

$$\lim_{n\to\infty} n = L \quad \text{for some } L \in \mathbb{R}.$$

Then from the ε-N definition of a limit (Definition 1.16) with $\varepsilon = 1$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \Longrightarrow \quad |n - L| < 1 \quad \Longleftrightarrow \quad L - 1 < n < L + 1 \quad \Longrightarrow \quad n < L + 1. \quad (10)$$

It follows from (10) for all $n \geq N$ we have $n < L + 1$. However, this would imply that $\lfloor L + 1 \rfloor$ is the largest natural number, which is a contradiction of the Archimedean Property. Therefore the sequence $a_n = n,\ n \in \mathbb{N}$, diverges.

C.1. $\boxed{\textit{Relevant page(s) in the notes: } 12}$

1.
$$\lim_{n\to\infty} \frac{7n^3 + 2n^2 - n + 16}{4n^3 + 98n^2 + 31n - 11} = \lim_{n\to\infty} \frac{7 + 2/n - 1/n^2 + 16/n^3}{4 + 98/n + 31/n^2 - 11/n^3} = \frac{7}{4}.$$

2. Let

$$a_n \equiv \frac{-1}{\sqrt[3]{19n + 42}} \quad \text{and} \quad c_n \equiv \frac{1}{\sqrt[3]{19n + 42}} \quad \text{and note that} \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = 0.$$

Since

$$a_n \leq \frac{(-1)^n}{\sqrt[3]{19n + 42}} \leq c_n$$

by the Sandwich Theorem (Theorem 1.23)

$$\lim_{n\to\infty} \frac{(-1)^n}{\sqrt[3]{19n + 42}} = 0.$$

3.

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{\sqrt{n}(\sqrt{n+1} - \sqrt{n})} &= \lim_{n\to\infty} \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n}(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})} \\
&= \lim_{n\to\infty} \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n}(n + 1 - n)} \\
&= \lim_{n\to\infty} \left(\sqrt{1 + 1/n} + 1\right) \\
&= 2.
\end{aligned}$$

C.2. $\boxed{\textit{Relevant page(s) in the notes: } \text{Supplementary Results}}$

Let

$$a_n \equiv \frac{974^n}{(n + 33)!}.$$

Then all $a_n \neq 0$ and

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{974^{n+1}}{(n+34)!} \times \frac{(n+33)!}{974^n} = \frac{974}{n+34} = \frac{974/n}{1+34/n}.$$

Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{974/n}{1+34/n}\right) = 0.$$

So by the Ratio Test for Sequences (Theorem S.1 of the Supplementary Results)

$$\lim_{n\to\infty}\frac{974^n}{(n+33)!} = 0.$$

C.3. $\boxed{\textit{Relevant page(s) in the notes: } 12}$

Define the sequences

$$a_n \equiv -\frac{1}{(n+4)^{1/39}}, \quad \text{and} \quad c_n \equiv \frac{1}{(n+4)^{1/39}} \quad n \in \mathbb{N}.$$

Then for all $n \in \mathbb{N}$,

$$-1 \le \cos(n) \le 1 \quad \Longrightarrow \quad -1 \le \cos^{703}(n) \le 1 \quad \Longrightarrow \quad a_n \le \frac{\cos^{703}(n)}{(n+4)^{1/39}} \le c_n. \quad (11)$$

Now

$$\lim_{n\to\infty} a_n = -\left\{\lim_{n\to\infty}\left(\frac{1}{n+4}\right)\right\}^{1/39} = 0^{1/39} = 0$$

and

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty}(-a_n) = -\lim_{n\to\infty}(a_n) = 0.$$

Because of (11) and the Sandwich Theorem (Theorem 1.23)

$$\lim_{n\to\infty}\frac{\cos^{703}(n)}{(n+4)^{1/39}} = 0.$$

C.5. $\boxed{\textit{Relevant page(s) in the notes: } 10}$

With division of the numerator and denominator by n^3 we have

$$\begin{aligned}\lim_{n\to\infty}\frac{(3n+8419)(7n+2768)(2n+3452)}{18n^3+93275} &= \lim_{n\to\infty}\frac{(3+8419/n)(7+2768/n)(2+3452/n)}{18+93275/n^3}\\ &= \frac{(3)(7)(2)}{18}\\ &= \frac{7}{3}.\end{aligned}$$

C.7. $\boxed{\textit{Relevant page(s) in the notes: } 10}$

Noting the identity

$$a^2 - b^2 = (a-b)(a+b) \quad \text{for all } a, b \in \mathbb{R}$$

we have, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\sqrt{2n^2+7n+3} - \sqrt{2n^2-4n+9} \\
&\quad = \frac{\left(\sqrt{2n^2+7n+3} - \sqrt{2n^2-4n+9}\right)\left(\sqrt{2n^2+7n+3} + \sqrt{2n^2-4n+9}\right)}{\sqrt{2n^2+7n+3} + \sqrt{2n^2-4n+9}} \\
&\quad = \frac{(2n^2+7n+3) - (2n^2-4n+9)}{\sqrt{2n^2+7n+3} + \sqrt{2n^2-4n+9}} \\
&\quad = \frac{11n-6}{\sqrt{2n^2+7n+3} + \sqrt{2n^2-4n+9}} \\
&\quad = \frac{11-6/n}{\sqrt{2+7/n+3/n^2} + \sqrt{2-4/n+9/n^2}}.
\end{aligned}$$

Hence

$$\begin{aligned}
\lim_{n\to\infty}\left(\sqrt{2n^2+7n+3} - \sqrt{2n^2-4n+9}\right) &= \lim_{n\to\infty}\left(\frac{11-6/n}{\sqrt{2+7/n+3/n^2} + \sqrt{2-4/n+9/n^2}}\right) \\
&= \frac{11}{2\sqrt{2}} \\
&= \frac{11\sqrt{2}}{4}.
\end{aligned}$$

C.11. $\boxed{\textit{Relevant page(s) in the notes: } \text{Supplementary Results}}$

Let

$$a_n \equiv \frac{(n+9)!17^{12n}}{(2n+3)!} \quad \text{for } n \in \mathbb{N}.$$

Then

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{(n+10)!17^{12(n+1)}}{(2n+5)!} \times \frac{(2n+3)!}{(n+9)!17^{12n}} \\
&= \frac{(n+10)17^{12}}{(2n+5)(2n+4)} \\
&= \frac{(1+10/n)}{(2+5/n)(2+4/n)} \times \frac{17^{12}}{n}.
\end{aligned}$$

Therefore,

$$L \equiv \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(1+10/n)}{(2+5/n)(2+4/n)} \times \lim_{n\to\infty}\frac{17^{12}}{n} = 0 < 1.$$

Since $L < 1$, by the Ratio Test for Sequences (Theorem S.1 of the Supplementary Results)

$$\lim_{n\to\infty}\frac{(n+9)!17^{12n}}{(2n+3)!} = 0.$$

C.12. $\boxed{\textit{Relevant page(s) in the notes:}\text{ Supplementary Results}}$

With division of the numerator and denominator by 3^n we have

$$\lim_{n\to\infty} \frac{(-2)^n + 3^n}{(-2)^{n+1} + 3^{n+1}} = \lim_{n\to\infty} \frac{(-2/3)^n + 1}{(-2)(-2/3)^n + 3}$$

Using the Ratio Test for Sequences (Theorem S.1 of the Supplementary Results)

$$\lim_{n\to\infty} (-2/3)^n = 0$$

we then obtain

$$\lim_{n\to\infty} \frac{(-2)^n + 3^n}{(-2)^{n+1} + 3^{n+1}} = \frac{1}{3}.$$

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