

CLASS 7; LIVE ϵ - δ PROOF; EXAMPLE 1

TO PROVE: $\lim_{x \rightarrow 4} \left(\frac{1}{2x-7} \right) = 1$

BACKGROUND WORKING TO OBTAIN δ

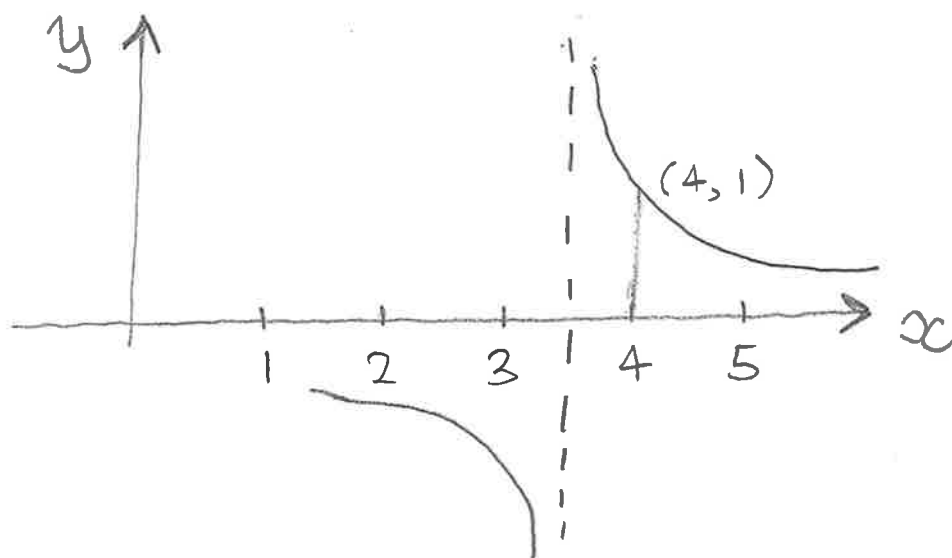
For each $\epsilon > 0$ we need to get to

$$\left| \frac{1}{2x-7} - 1 \right| < \epsilon \iff \left| \frac{1 - (2x-7)}{2x-7} \right| < \epsilon$$

$$\iff |x-4| \left(\frac{2}{|2x-7|} \right) < \epsilon$$

for all $x \neq 3\frac{1}{2}$.

The graph of $y = \frac{1}{2x-7}$ is



To avoid problems near the asymptote at $x = 3\frac{1}{2}$
we will insist on $\delta \leq \frac{1}{4}$ from now on.

Then $|x-4| < \delta$

$$\Rightarrow |x-4| < \frac{1}{4} \Leftrightarrow -\frac{1}{4} < x-4 < \frac{1}{4}$$

$$\Leftrightarrow \frac{15}{4} < x < \frac{17}{4}$$

$$\Leftrightarrow \frac{15}{2} < 2x < \frac{17}{2}$$

$$\Leftrightarrow \frac{1}{2} < 2x-7 < \frac{3}{2}$$

$$\Rightarrow |2x-7| > \frac{1}{2}$$

$$\Rightarrow \frac{1}{|2x-7|} < 2$$

$$\Rightarrow \left(\frac{2}{|2x-7|} \right) < 4 \dots\dots (\boxplus)$$

Recalling that we want to get to

$$|x-4| \left(\frac{2}{|2x-7|} \right) < \varepsilon, \dots\dots$$

adding the extra condition that

$$|x-4| < \varepsilon/4,$$

when combined with $(\boxplus)$ will get us home.

$$\text{Hence take } \delta = \min\left(\frac{1}{4}, \frac{\varepsilon}{4}\right).$$

FORMAL PROOF FOR EXAMPLE 1

For each $\varepsilon > 0$ let $\delta = \min(\frac{1}{4}, \frac{\varepsilon}{4})$.

Then $|x - 4| < \delta$ implies

$$|x - 4| < \frac{1}{4} \text{ and } |x - 4| < \frac{\varepsilon}{4} \dots (1)$$

The first inequality in (1) implies

$$\frac{15}{4} < x < \frac{17}{4} \Rightarrow \frac{1}{2} < 2x - 7 < \frac{3}{2} \Rightarrow |2x - 7| > \frac{1}{2}$$

$$\Rightarrow \frac{2}{|2x - 7|} < 4 \dots (2)$$

The second inequality in (1), when combined with (2) leads to

$$|x - 4| \left(\frac{2}{|2x - 7|} \right) < \frac{\varepsilon}{4} \times 4 = \varepsilon \dots (3)$$

Straightforward algebra leads to (3) being equivalent to

$$\left| \frac{1}{2x - 7} - 1 \right| < \varepsilon \text{ for all } x \neq 3\frac{1}{2}.$$

From the ε - δ definition of a limit,

$$\lim_{x \rightarrow 4} \left(\frac{1}{2x - 7} \right) = 1.$$

CLASS 7; LIVE ϵ - δ PROOF; EXAMPLE 2

TO PROVE: $\lim_{x \rightarrow 1} \left(\frac{\sqrt{x}}{\sqrt{x} + 1} \right) = \frac{1}{2}$

BACKGROUND WORKING TO OBTAIN δ

With $x \geq 0$, for each $\epsilon > 0$ we need to get to

$$\left| \frac{\sqrt{x}}{\sqrt{x} + 1} - \frac{1}{2} \right| < \epsilon \Leftrightarrow \left| \frac{2\sqrt{x} - (\sqrt{x} + 1)}{2(\sqrt{x} + 1)} \right| < \epsilon$$

$$\Leftrightarrow \frac{|\sqrt{x} - 1|}{2(\sqrt{x} + 1)} < \epsilon$$

$$\Leftrightarrow \frac{|\sqrt{x} - 1| |\sqrt{x} + 1|}{2(\sqrt{x} + 1) |\sqrt{x} + 1|} < \epsilon$$

$$\Leftrightarrow |x - 1| \left\{ \frac{1}{2(\sqrt{x} + 1)^2} \right\} < \epsilon \dots (\clubsuit)$$

We will start by insisting that $\delta \leq 1$. Then

$$|x - 1| < \delta \Rightarrow |x - 1| < 1 \Rightarrow 0 < x < 2$$

$$\Rightarrow 1 < \sqrt{x} + 1 < \sqrt{2} + 1$$

$$\Rightarrow (\sqrt{x} + 1)^2 > 1 \Rightarrow \frac{1}{2(\sqrt{x} + 1)^2} < \frac{1}{2} \dots (\diamond)$$

Recalling that we want to get to $(\clubsuit)$ and noting $(\diamond)$, we should also have $|x - 1| < 2\epsilon$.

Hence $\delta = \min(1, 2\epsilon)$ looks good.

FORMAL PROOF FOR EXAMPLE 2

For each $\epsilon > 0$ let $\delta = \min(1, 2\epsilon)$.

Then $|x-1| < \delta$ implies

$$|x-1| < 1 \quad \text{and} \quad |x-1| < 2\epsilon \quad \dots (4)$$

The first inequality in (4), implies

$$0 < x < 2 \Rightarrow 0 < \sqrt{x} < \sqrt{2} \Rightarrow 1 < \sqrt{x} + 1 < \sqrt{2} + 1$$

$$\Rightarrow |\sqrt{x} + 1| > 1 \Rightarrow \frac{1}{(\sqrt{x} + 1)^2} < 1$$

$$\Rightarrow \frac{1}{2(\sqrt{x} + 1)^2} < \frac{1}{2} \quad \dots (5)$$

The second inequality in (4), when combined with (5) leads to

$$|x-1| \left\{ \frac{1}{2(\sqrt{x} + 1)^2} \right\} < 2\epsilon \times \frac{1}{2} = \epsilon \quad \dots (6)$$

Straightforward algebra leads to (6) being equivalent to $\left| \frac{\sqrt{x}}{\sqrt{x} + 1} - \frac{1}{2} \right| < \epsilon$ for all $x \geq 0$.

From the ϵ - δ definition of a limit,

$$\lim_{x \rightarrow 1} \left(\frac{\sqrt{x}}{\sqrt{x} + 1} \right) = \frac{1}{2}.$$